MVE165/MMG631
Linear and integer optimization with applications
Lecture 5
Linear programming duality and sensitivity analysis

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A linear program with optimal value z^*

$$
\begin{array}{rrrlll}
z^* = \max \quad z := & 20x_1 & +18x_2 & & & \text{weights} \\
\text{subject to} & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
 & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
 & & x_1, x_2 & \geq 0 & &
\end{array}
$$

[DRAW GRAPH]

- What is the largest possible value of z (i.e., z^*)?

Compute upper estimates of z^*, e.g.:

- Multiply (1) by 3:

  $\Rightarrow \; 21x_1 + 30x_2 \leq 10800 \qquad \Rightarrow \; z^* \leq 10800$
- Multiply (2) by 1.5:

  $\Rightarrow \; 24x_1 + 18x_2 \leq 8100 \qquad \Rightarrow \; z^* \leq 8100$
- Combine: $0.6\times(1)+1\times(2)$:

  $\Rightarrow \; 20.2x_1 + 18x_2 \leq 7560 \qquad \Rightarrow \; z^* \leq 7560$

An "intuitive" derivation of duality, II

A linear program with optimal value z^*

$$\begin{array}{llrrrcc}
\text{maximize} & z := & 20x_1 & +18x_2 & & & \text{weights} \\
\text{subject to} & & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
 & & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
 & & & x_1, x_2 & \geq 0 & &
\end{array}$$

[DRAW GRAPH]

- Do better than guess—compute *optimal* weights!
- Value of estimate: $w = 3600v_1 + 5400v_2 \rightarrow \min$

Constraints on the weights

$$\begin{array}{rl}
7v_1 + 16v_2 & \geq 20 \\
10v_1 + 12v_2 & \geq 18 \\
v_1, v_2 & \geq 0
\end{array}$$

An "intuitive" derivation of duality, III (Ch. 6.1)

The best (lowest) possible upper estimate of z^*

$$\begin{array}{rrcrcrl} \text{minimize} & w := & 3600v_1 & + & 5400v_2 & & \\ \text{subject to} & & 7v_1 & + & 16v_2 & \geq & 20 \\ & & 10v_1 & + & 12v_2 & \geq & 18 \\ & & & & v_1, v_2 & \geq & 0 \end{array}$$

- A linear program! [Draw graph!!]
- It is called the *linear programming dual* of the original linear program

The lego model – the market problem

Consider the lego problem

$$\begin{array}{llrcrcl} \text{maximize} & z = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & 2x_1 & + & x_2 & \leq & 6 \\ & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

- Option: Sell bricks instead of making furniture
- $v_1(v_2) =$ price of a large (small) brick
- The market wishes to *minimize the payment:* $\min\ 6v_1 + 8v_2$

Sell only if prices are high enough

- $2v_1 + 2v_2 \geq 1600$ – otherwise better to make tables
- $v_1 + 2v_2 \geq 1000$ – otherwise better to make chairs
- $v_1, v_2 \geq 0$ – don't sell at a negative price

A general linear program on "standard form"

A linear program with n non-negative variables, m equality constraints ($m < n$), and non-negative right-hand-sides

$$
\begin{aligned}
\text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m, \\
& x_j \geq 0, \quad j = 1, \ldots, n,
\end{aligned}
$$

where

$$
\begin{aligned}
& x_j \in \mathbb{R}, \quad j = 1, \ldots, n, \\
& a_{ij} \in \mathbb{R}, \quad i = 1, \ldots, m, \\
& \qquad\qquad j = 1, \ldots, n, \\
& b_i \geq 0, \quad i = 1, \ldots, m, \\
& c_j \in \mathbb{R}, \quad j = 1, \ldots, n.
\end{aligned}
$$

Or, on matrix form

$$
\begin{aligned}
\text{maximize} \quad & z = \mathbf{c}^{\mathrm{T}} \mathbf{x}, \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}^n,
\end{aligned}
$$

where

$\mathbf{x} \in \mathbb{R}^n$,
$\mathbf{A} \in \mathbb{R}^{m \times n}$,
$\mathbf{b} \in \mathbb{R}^m_+$,
$\mathbf{c} \in \mathbb{R}^n$.

Linear programming duality

To each primal linear program corresponds a dual linear program

(Primal)

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}^n \end{aligned}$$

(Dual)

$$\begin{aligned} \text{maximize} \quad & w = \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c} \end{aligned}$$

The component forms of the primal and dual programs

(Primal)

$$\begin{aligned} \min \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

(Dual)

$$\begin{aligned} \max \quad & w = \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \\ & j = 1, \dots, n \end{aligned}$$

An example of linear programming duality

A primal linear program

$$
\begin{array}{rrrrll}
\text{minimize} & z = & 2x_1 & +3x_2 & & \\
\text{subject to} & & 3x_1 & +2x_2 & = 14 & \\
& & 2x_1 & -4x_2 & \geq 2 & \\
& & 4x_1 & +3x_2 & \leq 19 & \\
& & & x_1, x_2 & \geq 0 &
\end{array}
\qquad
\begin{array}{c}
\text{weights/duals} \\
y_1 \\
y_2 \\
y_3 \\
\
\end{array}
$$

The corresponding dual linear program

$$
\begin{array}{rrrrrll}
\text{maximize} & w = & 14y_1 & +2y_2 & +19y_3 & & \\
\text{subject to} & & 3y_1 & +2y_2 & +4y_3 & \leq 2 & \\
& & 2y_1 & -4y_2 & +3y_3 & \leq 3 & \\
& & y_1 & & & \text{free} & \\
& & & y_2 & & \geq 0 & \\
& & & & y_3 & \leq 0 &
\end{array}
\qquad
\begin{array}{c}
\text{weights/primals} \\
x_1 \\
x_2 \\
\ \\
\ \\
\
\end{array}
$$

Rules for constructing the dual program (Ch. 6.2)

maximization $\Longleftrightarrow$ minimization

dual program	$\Longleftrightarrow$	primal program
primal program	$\Longleftrightarrow$	dual program
constraints	$\Longleftrightarrow$	*variables*
$\geq$	$\Longleftrightarrow$	≤ 0
$\leq$	$\Longleftrightarrow$	≥ 0
$=$	$\Longleftrightarrow$	free
variables	$\Longleftrightarrow$	*constraints*
≥ 0	$\Longleftrightarrow$	$\geq$
≤ 0	$\Longleftrightarrow$	$\leq$
free	$\Longleftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

Primal			Dual		
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	(1a)	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$	(2a)
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	(1b)	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$	(2b)
	$\mathbf{x} \geq \mathbf{0}^n$	(1c)			

Weak duality [Th. 6.1]

Let $\mathbf{x}$ be a feasible point in the primal (minimization) and $\mathbf{y}$ be a feasible point in the dual (maximization). Then, it holds that

$$z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y} = w$$

Proof: $$z \underbrace{=}_{(1a)} \mathbf{c}^{\mathrm{T}}\mathbf{x} \underbrace{\geq}_{(2b),\ (1c)} \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x} \underbrace{=}_{(1b)} \mathbf{y}^{\mathrm{T}}\mathbf{b} \underbrace{=}_{(2a)} w. \quad \square$$

In the course book, the primal is formulated with inequality constraints in (1b): adjust the dual and the proof for that case!

Duality properties, II (Ch. 6.3)

Primal	Dual
minimize $z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	maximize $w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$
subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$	subject to $\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}^n$	

Strong duality [Th. 6.3]

In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.

Primal		Dual	
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}^n$		

Complementary slackness [Th. 6.5; proof in the course book]

If $\mathbf{x}$ is *optimal in the primal* and $\mathbf{y}$ is *optimal in the dual*, then it holds that

$$\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0.$$

If $\mathbf{x}$ is *feasible in the primal*, $\mathbf{y}$ is *feasible in the dual*, and $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$, then

$\mathbf{x}$ and $\mathbf{y}$ are optimal in their respective problems.

Duality properties, IV (Ch. 6.3)

Primal

$$z^* := \min z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N$$
$$\text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$
$$\mathbf{x}_B \geq \mathbf{0}^m, \mathbf{x}_N \geq \mathbf{0}^{n-m}$$

Dual

$$w^* := \max w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$$
$$\text{subject to} \quad \mathbf{B}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}_B$$
$$\mathbf{N}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}_N$$

Duality theorem [Th. 6.4]

Assume that $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ is an optimal basic (feasible) solution to the primal problem. Then $\mathbf{y}^{\mathrm{T}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ is an optimal solution to the dual problem and $z^* = w^*$.

Proof structure [full proof in the course book]

1. $\mathbf{y}^{\mathrm{T}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ is feasible in the dual problem
2. The optimal objective values z^* and w^* are equal
3. Follows from complementarity: $(\mathbf{x}_B, \mathbf{x}_N)$ and $\mathbf{y}$ are feasible in the primal and dual respective problem and $z^* = w^*$

Relations between primal and dual optimal solutions

primal (dual) problem	$\iff$	dual (primal) problem
unique and non-degenerate solution	$\iff$	unique and non-degenerate solution
unbounded solution	$\Longrightarrow$	no feasible solutions
no feasible solutions	$\Longrightarrow$	unbounded solution *or* no feasible solutions
degenerate solution	$\iff$	alternative solutions

Homework

Exercises on linear programming duality

- Formulate and solve graphically the dual of:

$$
\begin{array}{lrrrrl}
\text{minimize} & z = & 6x_1 & +3x_2 & +x_3 & \\
\text{subject to} & & 6x_1 & -3x_2 & +x_3 & \geq 2 \\
 & & 3x_1 & +4x_2 & +x_3 & \geq 5 \\
 & & & & x_1, x_2, x_3 & \geq 0
\end{array}
$$

- Then find the optimal primal solution
- Verify that the dual of the dual equals the primal

Derivation of the simplex method (review) (Ch. 4.8)

- B = index set of basic var's, N = index set of non-basic var's

$\Rightarrow$ $|B| = m$ and $|N| = n - m$

- Partition matrix/vectors: $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- The matrix $\mathbf{B}$ ($\mathbf{N}$) contains the columns of $\mathbf{A}$ corresponding to the index set B (N) — Analogously for $\mathbf{x}$ and $\mathbf{c}$

Original linear program

$$\begin{aligned} \text{minimize } z &= \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to } \quad \mathbf{A}\mathbf{x} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n \end{aligned}$$

Rewritten linear program

$$\begin{aligned} \text{minimize } z &= \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N \\ \text{subject to } \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N &= \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \ \mathbf{x}_N &\geq \mathbf{0}^{n-m} \end{aligned}$$

Substitute: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \Longrightarrow$

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Optimality and feasibility (review)

Optimality condition (for minimization)

The basis B is *optimal* if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
(marginal values = reduced costs ≥ 0)

If not, choose as *entering* variable $j \in N$ the one with the lowest (negative) value of the reduced cost $c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$

Feasibility condition

For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

Choose the *leaving* variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \;\middle|\; (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

In the simplex tableau, we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

- $\mathbf{s}$ denotes possible slack variables [columns for $\mathbf{s}$ are copies of certain columns for $(\mathbf{x}_B, \mathbf{x}_N)$]
- The computations performed by the simplex algorithm involve matrix inversions (i.e., $\mathbf{B}^{-1}$) and *updates* of these
- A non-basic (basic) variable enters (leaves) the basis $\Rightarrow$ one column, $\mathbf{A}_j$, in $\mathbf{B}$ is replaced by another, $\mathbf{A}_k$
- Row operations $\Leftrightarrow$ Updates of $\mathbf{B}^{-1}$ (and of $\mathbf{B}^{-1}\mathbf{N}$, $\mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$)

$\Rightarrow$ Efficient numerical computations are crucial for the performance of the simplex algorithm

Sensitivity analysis—changes in the optimal solution as functions of changes in the problem data (Ch. 5)

- How does the optimum change when the *right-hand-sides* (resources, e.g.) *change*?
- When the *objective coefficients* (prices, e.g.) *change*?

Assume that the basis B is optimal:

$$
\begin{array}{ll}
\text{minimize} & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\
\text{subject to} & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\
 & \mathbf{x}_N \geq \mathbf{0}^{n-m},
\end{array}
$$

where $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$

Changes in the right-hand-side coefficients

Shadow price = dual price [Def. 5.3]

The *shadow price* of a constraint is defined as the change in the optimal value as a function of the (marginal) change in the RHS. It equals the optimal value of the corresponding dual variable.
In AMPL: display constraint_name.dual

- Suppose $\mathbf{b}$ changes to $\mathbf{b} + \Delta\mathbf{b}$

$\Rightarrow$ New optimal value:

$$\boxed{z^{\text{new}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = z + \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\Delta\mathbf{b}}$$

- The current basis is feasible if $\boxed{\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0}$
- If not: negative values will occur in the RHS of the simplex tableau
- The reduced costs are unchanged (positive, at optimum) $\Rightarrow$ this can be resolved using the *dual simplex method*

Changes in the right-hand-side coefficients

A linear program

$$\begin{array}{lrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 \\ & & x_1 & & \leq 3 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

DRAW GRAPH

The optimal solution is given by

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	13
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Changes in the right-hand-side coefficients

Change the right-hand-side according to

$$\begin{array}{lrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 + \delta \\ & & x_1 & & \leq 3 \\ & & x_1, x_2 & \geq 0 & \end{array}$$

The change in the RHS is given by $\mathbf{B}^{-1}(0, \delta, 0)^{\mathrm{T}} = (\frac{1}{2}\delta, 0, -\frac{1}{2}\delta)^{\mathrm{T}}$ $\Rightarrow$ *new optimal tableau:*

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	$13 + \delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$5 + \frac{1}{2}\delta$
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$3 - \frac{1}{2}\delta$

- *The current basis is feasible if $-10 \leq \delta \leq 6$ (i.e., if RHS ≥ 0)*
- In AMPL: `display constraint_name.down, .current, .up`

Suppose $\delta = 8$. The simplex tableau then appears as

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	21
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	9
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	-1

- Dual simplex iteration: $s_1 = -1$ has to leave the basis
- Find smallest ratio between reduced cost (non-basic column) and (negative) elements in the "s_1-row" (to stay optimal)

s_2 will enter the basis — *new optimal* tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	2	0	5	19
x_2	0	0	1	1	0	2	8
x_1	0	1	0	0	0	1	3
s_2	0	0	0	-2	1	-3	2

Changes in the objective coefficients

Reduced cost

The *reduced cost* of a non-basic variable defines the change in the objective value when the value of the corresponding variable is (marginally) increased.
The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$ (i.e., marginal values = reduced costs ≥ 0)
In AMPL: display variable_name.rc

- Suppose $\mathbf{c}$ changes to $\mathbf{c} + \Delta\mathbf{c}$
- The new optimal value:

$$\boxed{z^{\mathrm{new}} = (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} = z + \Delta\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}}$$

- The current basis is optimal if

$$\boxed{(\mathbf{c}_N + \Delta\mathbf{c}_N)^{\mathrm{T}} - (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}}$$

- If not: more simplex iterations to find the optimal solution

Changes in the objective coefficients

Change the objective according to

$$\begin{array}{llrrl} \text{minimize} & z = & -x_1 & +(-2+\delta)x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 \\ & & x_1 & & \leq 3 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

The changes in the reduced costs are given by $-(\delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} = (-\frac{1}{2}\delta, -\frac{1}{2}\delta) \Rightarrow$ *new optimal tableau:*

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	$1-\frac{1}{2}\delta$	$2-\frac{1}{2}\delta$	$13-5\delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- *The current basis is optimal if $\delta \leq 2$ (i.e., if reduced costs ≥ 0)*
- In AMPL: display variable_name.down, .current, .up

Changes in the objective coefficients

Suppose $\delta = 4 \Rightarrow$ new tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	-1	0	-7
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Let s_2 enter and x_2 leave the basis. New optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	2	0	0	1	3
s_2	0	0	2	0	1	1	10
x_1	0	1	0	0	0	1	3
s_1	0	0	1	1	0	2	8