

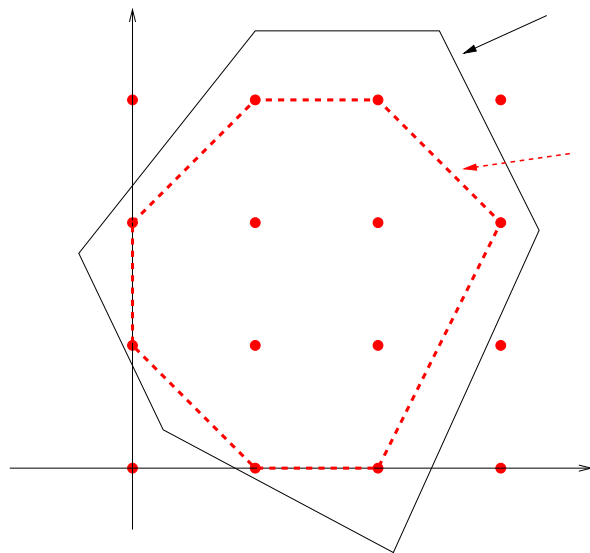
MVE165/MMG631  
Linear and integer optimization with applications  
Lecture 8  
Discrete optimization: theory and algorithms

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- Relaxations: cutting planes and Lagrangean duals
- TSP and routing problems
- Branch-and-bound for structured problems

$$Ax \leq b$$



**Ideal** since all extreme points are integral

The linear program has **integer extreme points**

# Cutting planes: A very small example

Consider the following ILP:

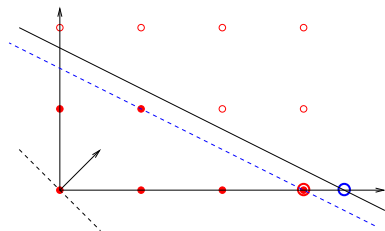
$$\min\{-x_1 - x_2 : 2x_1 + 4x_2 \leq 7, x_1, x_2 \geq 0 \text{ and integer}\}$$

- ILP optimal solution:  $z = -3, \mathbf{x} = (3, 0)$
- LP (continuous relaxation) optimum:  $z = -3.5, \mathbf{x} = (3.5, 0)$

Generate a simple cut

*"Divide the constraint" by 2  
and round the RHS down*  
 $x_1 + 2x_2 \leq 3.5 \Rightarrow x_1 + 2x_2 \leq 3$

Adding this cut to the  
continuous relaxation yields  
the optimal ILP solution



Consider the ILP

$$\begin{array}{llll} \max & 7x_1 + 10x_2 & & \\ \text{subject to} & -x_1 + 3x_2 \leq 6 & (1) & \\ & 7x_1 + x_2 \leq 35 & (2) & \\ & x_1, x_2 \geq 0, \text{ integer} & & \end{array}$$

- LP optimum:  $z = 66.5$ ,  $\mathbf{x} = (4.5, 3.5)$
- ILP optimum:  $z = 58$ ,  $\mathbf{x} = (4, 3)$

Generate a VI:

“Add” the two constraints (1) and

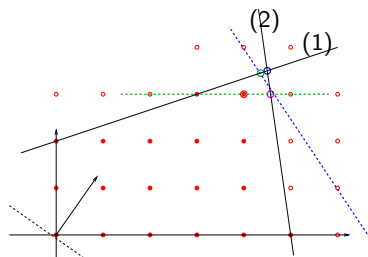
(2):  $6x_1 + 4x_2 \leq 41 \Rightarrow$

$3x_1 + 2x_2 \leq 20 \Rightarrow \mathbf{x} = (4.36, 3.45)$

Generate another VI:

“ $7 \cdot (1) + (2)$ ”:  $22x_2 \leq 77 \Rightarrow x_2 \leq 3$

$\Rightarrow \mathbf{x} = (4.57, 3)$



# Cutting plane algorithms (iteratively better approximations of the convex hull) (Ch. 14.5)

- Choose a suitable mathematical formulation of the problem

## A general cutting plane algorithm

- 1 Solve the linear programming (LP) relaxation
- 2 If the LP solution is integer: stop; an optimal solution to ILP is found
- 3 Add one or several *valid inequalities* that *cut off the fractional* solution but *none of the integer solutions*
- 4 Resolve the new problem and go to step 2.

- *Remark:* An inequality in higher dimensions defines a *hyper-plane*; therefore the name cutting *plane*

# About cutting plane algorithms

- Problem: It may be necessary to generate VERY MANY cuts
- Each cut should also pass through at least one integer point  
⇒ faster convergence
- Methods for generating valid inequalities
  - Chvatal-Gomory cuts (combine constraints, make beneficial roundings of LHS and RHS)
  - Gomory's method: generate cuts from an optimal simplex basis (Ch. 14.5.1)
- Pure cutting plane algorithms are usually less efficient than branch-&-bound
- In commercial solvers (e.g. CPLEX), cuts are used to help (presolve) the branch-&-bound algorithm
- For problems with specific structures (e.g. TSP and set covering) problem specific classes of cuts are used

# Lagrangian relaxation ( $\Rightarrow$ optimistic estimates of $z^*$ ) (Ch. 17.1–17.2)

Consider a minimization integer linear program (ILP)

$$\begin{aligned} \text{[ILP]} \quad z^* = \quad & \min \quad \mathbf{c}^T \mathbf{x} \\ & \text{subject to} \quad \mathbf{Ax} \leq \mathbf{b} & (1) \\ & \quad \quad \quad \mathbf{Dx} \leq \mathbf{d} & (2) \\ & \quad \quad \quad \mathbf{x} \geq \mathbf{0} \text{ and integer} \end{aligned}$$

Assume that the constraints (1) are complicating (subtour eliminating constraints for TSP, e.g.)

- Define the set  $X = \{\mathbf{x} \in Z_+^n \mid \mathbf{Dx} \leq \mathbf{d}\}$
- Remove the constraints (1) and add them—with penalty parameters  $\mathbf{v}$ —to the objective function

$$h(\mathbf{v}) = \min_{\mathbf{x} \in X} \{\mathbf{c}^T \mathbf{x} + \mathbf{v}^T (\mathbf{Ax} - \mathbf{b})\} \quad (3)$$



# Weak duality of Lagrangian relaxations

## Theorem

For any  $\mathbf{v} \geq \mathbf{0}$  it holds that  $h(\mathbf{v}) \leq z^*$ .

## Bevis.

Let  $\bar{\mathbf{x}}$  be feasible in [ILP]  $\Rightarrow \bar{\mathbf{x}} \in X$  and  $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}$ . It then holds that

$$h(\mathbf{v}) = \min_{\mathbf{x} \in X} \{ \mathbf{c}^T \mathbf{x} + \mathbf{v}^T (\mathbf{A}\mathbf{x} - \mathbf{b}) \} \leq \mathbf{c}^T \bar{\mathbf{x}} + \mathbf{v}^T (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) \leq \mathbf{c}^T \bar{\mathbf{x}}.$$

Since an optimal solution  $\mathbf{x}^*$  to [ILP] is also feasible, it holds that

$$h(\mathbf{v}) \leq \mathbf{c}^T \mathbf{x}^* = z^*. \quad \square$$

$\Rightarrow h(\mathbf{v})$  is a *lower bound* on the optimal value  $z^*$  for any  $\mathbf{v} \geq \mathbf{0}$

The best lower bound is given by

$$h^* = \max_{\mathbf{v} \geq \mathbf{0}} h(\mathbf{v}) = \max_{\mathbf{v} \geq \mathbf{0}} \left\{ \min_{\mathbf{x} \in X} \{ \mathbf{c}^T \mathbf{x} + \mathbf{v}^T (\mathbf{A}\mathbf{x} - \mathbf{b}) \} \right\} \leq z^*$$

# Tractable Lagrangian relaxations

- Special algorithms for **maximizing** the Lagrangian dual function  $h$  exist (e.g., subgradient optimization, Ch. 17.3)
- $h$  is always **concave** but typically **nondifferentiable**
- For each value of  $\mathbf{v}$  chosen, a **subproblem** (3) must be solved
- For general ILP's: typically a non-zero **duality gap**  $h^* < z^*$
- The Lagrangian relaxation bound is never worse than the linear programming relaxation bound, i.e.  $z^{\text{LP}} \leq h^* \leq z^*$
- If the set  $X$  has the **integrality property** (i.e.,  $X^{\text{LP}}$  has integral extreme points) then  $h^* = z^{\text{LP}}$
- Choose the constraints ( $\mathbf{Ax} \leq \mathbf{b}$ ) to dualize such that the relaxed problem (3) is **computationally tractable** but still does **not** possess the integrality property

## [HOMEWORK]

Find optimistic and pessimistic bounds for the following ILP example using the branch-&-bound algorithm, a cutting plane algorithm, and Lagrangean relaxation.

$$\begin{array}{ll} \max & 5x_1 + 4x_2 \\ \text{s.t.} & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{array}$$

The linear programming optimal solution is given by  $z = 23.75$ ,  $x_1 = 3.75$  and  $x_2 = 1.25$

# ILP formulation of the TSP problem

- $d_{ij}$ : distance from city  $i$  to city  $j$
- Binary variables  $x_{ij}$  for each connection
- $V = \{1, \dots, n\}$ : set of nodes (cities)

$$\min \sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij}, \quad (0)$$

$$\text{s.t.} \quad \sum_{j \in V} x_{ij} = 1, \quad i \in V, \quad (1)$$

$$\sum_{i \in V} x_{ij} = 1, \quad j \in V, \quad (2)$$

$$\sum_{i \in U, j \in V \setminus U} x_{ij} \geq 1, \quad \forall U \subset V : 2 \leq |U| \leq |V| - 2, \quad (3)$$

$$x_{ij} \in \{0, 1\}, \quad i, j \in V \quad (4)$$

- (0)–(2), (4): assignment problem
- Enter and leave each city exactly once  $\Leftrightarrow$  (1) and (2)
- Constraints (3): **subtour elimination**

# Solution methods for the TSP Problem

- Tailored branch-&-bound (Ch. 15)
  - Heuristics
    - Constructive heuristics (Ch. 16.3)
    - Local search heuristics (Ch. 16.4)
    - Approximation algorithms (Ch. 16.6)
    - Metaheuristics (Ch. 16.5)
  - Lagrangean relaxation and subgradient optimization (Ch. 17).
  - Common difficulty for **all** solution methods for the TSP:  
**Combinatorial explosion**: # possible tours  $\approx n!$
- ⇒ Very many subtour elimination constraints (3)

- Relaxing just the binary constraints (4) in TSP does not yield a tractable problem, since the number of subtour elimination constraints (3) is very large
- ⇒ An LP with **very many** constraints
- Relaxing the subtour eliminating constraints (3) yields an assignment problem, which can be solved in polynomial time
  - Solutions to a relaxed problem typically contains a number of sub-tours
  - Branch on these sub-tours (rather than on fractional variables)
  - Branching  $\Leftrightarrow$  partitioning of the solution space

DRAW AN EXAMPLE