MVE165/MMG631
Linear and integer optimization with applications
Lecture 11
Shortest paths; dynamic programming; linear programming formulations of network flows

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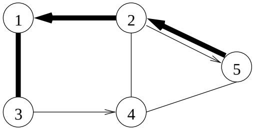
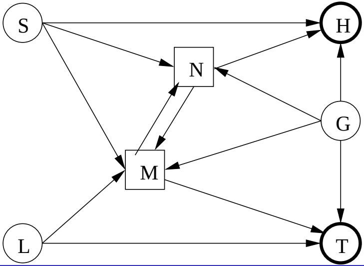
Flows in networks, in particular shortest paths

A path from node 5 *to node* 3



A flow network

- Supply nodes: S, G, L
- Demand nodes: H, T
- Storage (intermediate): M, N
- Limited capacities on links
- Minimize costs for transport and storage



Many different problems can be formulated as graph or network flow models

- Find the total capacity of a given water pipeline network
- Find a time schedule (starting and completion times) for the activities in a project
- How much goods should be transported from each supplier to each point of demand in a transportation system, and which links should be used to what extent
- Find the optimal set of maintenance occasions over a time period for *one* component

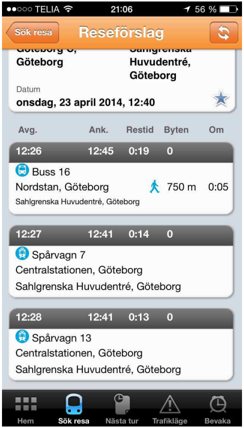
A useful application

Question:

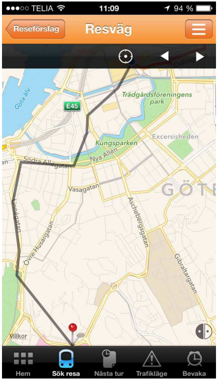
In terms of networks

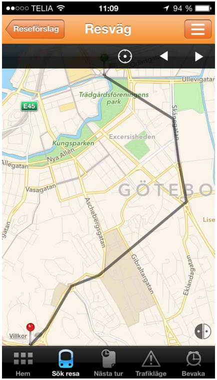
- What question do we ask?
- Discuss with your neigbour!
- Suggestions?

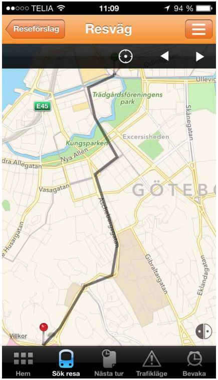
Answer:

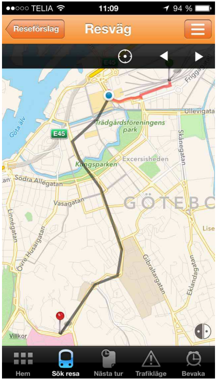


The shortest path problem: a useful application







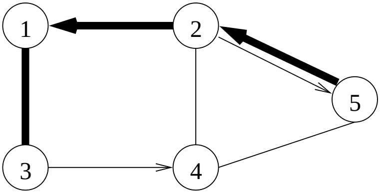


A number of "short" (or fast) paths that

- depart at the earliest "now", and
- arrive at the latest "around 12:40"

Shortest path problem—properties & solution

- What properties of the problem can be utilized to construct an efficient solution method for the shortest path problem?



- Discuss
- *... draw on the board ...*

Solving the problem ...

- How long is the shortest path from 1 to 6? Why?
- Discuss

- How can we find this path, using the "spatial" properties of the network?
- Discuss

- *... adjust spatially the illustration on the board ...*

A mathematical model

Let y_i = length of the shortest path from node 1 to node i

- *"Stretch the arcs"* between the nodes 1 and 6 $\Leftrightarrow$ maximize the difference of the "potentials" y_6 and y_1:

$$(y_6 - y_1) \longrightarrow \max$$

- The arcs are not elastic:

$$\begin{aligned} y_2 - y_1 &\leq 4 & \qquad y_4 - y_2 &\leq 2 & \qquad y_5 - y_4 &\leq 5 \\ y_3 - y_1 &\leq 2 & \qquad y_4 - y_3 &\leq 3 & \qquad y_6 - y_4 &\leq 3 \\ y_2 - y_3 &\leq 3 & \qquad y_5 - y_2 &\leq 5 & \qquad y_6 - y_5 &\leq 1 \end{aligned}$$

- A system of nine inequalities (*not* equations) and six unknowns,
- An objective function to be maximized
- All involved functions are differences between potentials

Another mathematical model—based on flows

Send one unit of flow along the shortest path from node 1 to node 6

- Let $x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the shortest path from 1 to 6} \\ 0 & \text{otherwise} \end{cases}$
- Objective:
  $(4x_{12} + 2x_{13} + 3x_{32} + 2x_{24} + 3x_{34} + 5x_{25} + 5x_{45} + 3x_{46} + 1x_{56}) \to \min$
- Node balance (any flow that enters a node must also leave it)

$$
\begin{array}{rrrrrrrrrcr}
-x_{12} & -x_{13} & & & & & & & & = & -1 \\
+x_{12} & & +x_{32} & -x_{24} & & -x_{25} & & & & = & 0 \\
 & +x_{13} & -x_{32} & & -x_{34} & & & & & = & 0 \\
 & & & +x_{24} & +x_{34} & & -x_{45} & -x_{46} & & = & 0 \\
 & & & & & +x_{25} & +x_{45} & & -x_{56} & = & 0 \\
 & & & & & & & +x_{46} & +x_{56} & = & 1 \\
x_{12}\,, & x_{13}\,, & x_{32}\,, & x_{24}\,, & x_{34}\,, & x_{25}\,, & x_{45}\,, & x_{46}\,, & x_{56} & \geq & 0
\end{array}
$$

The mathematical models are each other's LP duals

The optimal solutions to the two models

- $y_1^* = 0,\ y_2^* = 4,\ y_3^* = 2,\ y_4^* = 5,\ y_5^* = 9,\ y_6^* = 8$
- $\Leftrightarrow$ maximal difference of the potentials:

$$(y_6^* - y_1^*) = 8$$

- Fulfilment of the constraints:

$$\begin{array}{lll} y_2^* - y_1^* = 4 = 4 & y_4^* - y_2^* = 1 < 2 & y_5^* - y_4^* = 4 < 5 \\ y_3^* - y_1^* = 2 = 2 & y_4^* - y_3^* = 3 = 3 & y_6^* - y_4^* = 3 = 3 \\ y_2^* - y_3^* = 2 < 3 & y_5^* - y_2^* = 5 = 5 & y_6^* - y_5^* = -1 < 1 \end{array}$$

- The optimal solution to the flow model:

$$x_{13}^* = x_{34}^* = x_{46}^* = 1$$

$$x_{12}^* = x_{32}^* = x_{24}^* = x_{25}^* = x_{45}^* = x_{56}^* = 0$$

[Illustrate the complementarity]

A general LP formulation: shortest path from node $s \in N$ to node $t \in N$ in a directed graph $G = (N, A, \mathbf{d})$

- For each $(i, j) \in A$, let x_{ij} be the flow on arc (i, j)
- *Flow balance in each node $k \in N$*
- $x_{ij} = 1$ if arc (i, j) is in the shortest path; $x_{ij} = 0$ otherwise

Linear programming formulation:

$$\begin{array}{lrcl}
\min & \displaystyle\sum_{(i,j)\in A} d_{ij}x_{ij}, & & \\
\text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & \begin{cases} -1, & k = s, \\ 1, & k = t, \\ 0, & k \in N \setminus \{s, t\}, \end{cases} \\
& x_{ij} & \geq & 0, \quad (i, j) \in A
\end{array}$$

Linear programming dual:

$$\begin{array}{lrcll}
\max & y_t - y_s, & & & \\
\text{s.t.} & y_j - y_i & \leq & d_{ij}, & (i, j) \in A \\
& y_k & & \text{free}, & k \in N
\end{array}$$

The shortest path problem (Ch. 8.4)

- Given: a network/graph of nodes N, (directed) arcs A, and arc lengths d_{ij}, $(i,j) \in A$
- Denoted $G = (N, A, \mathbf{d})$
- Find the shortest path from a source node ($s \in N$) to a destination node ($t \in N$)
- Important concept: For $i_\ell \in N$, $\ell \in \{1, \dots, k\}$, a *negative cycle* $\{i_1, i_2, \dots, i_k\}$ is such that

$$\sum_{\ell=1}^{k-1} d_{i_\ell i_{\ell+1}} + d_{i_k i_1} < 0$$

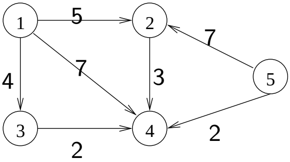
What impact does this have on a shortest path?

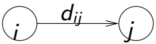
Principle of optimality formulated by Bellman's equations (Ch. 8.4.1)

- In a graph with *no negative cycles*, optimal paths have optimal subpaths
- A shortest path from node s node to t that passes through node k contains a shortest path from node s to node k
- Let y_j denote the length of the shortest path from node s to j

Bellman's equations:

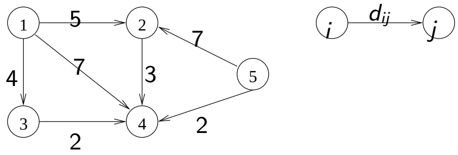
- $y_s = 0$
- $y_j = \min_i \left\{ y_i + d_{ij} : (i,j) \in A \right\}$ for all $j \neq s$





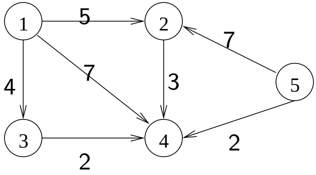
Solution method I: Bellman's equations
A special case of dynamic programming

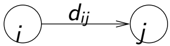
- If the graph is directed with no cycles: solve Bellman's equations in topological order
- Shortest path from node 1 to each of the other nodes (1,5,2,3,4):
  - $y_1 := 0$
  - $y_5 := \min\{\infty\} = \infty$
  - $y_2 := \min\{\infty; y_1 + d_{12}; y_5 + d_{52}\} = \min\{\infty; 0+5; \infty\} = 5$
  - $y_3 := \min\{\infty; y_1 + d_{13}\} = \min\{\infty; 0+4\} = 4$
  - $y_4 := \min\{\infty; y_1 + d_{14}; y_2 + d_{24}; y_3 + d_{34}; y_5 + d_{54}\} = \min\{\infty; 0+7; 5+3; 4+2; \infty+2\} = 6$



Solution method II: Dijkstra's algorithm

- The graph may contain cycles but all arc costs must be non-negative (i.e., $d_{ij} \geq 0$)





- *Solve the example on the board*

Algorithms for the shortest path problem: Dijkstra (Ch.8.4.2)

- Find the shortest path between node s and node i when all arc lengths are non-negative (cycles may exist)
- $N =$ set of all nodes; source node $s \in N$
- $d_{ij} =$ length of arc from i to j for all $i, j \in N$
- $d_{ij} := \infty$ if no direct arc from i to j

Dijkstra's shortest path algorithm

Step 0: $S := \{s\}$, $\bar{S} := N \setminus \{s\}$, and $y_i := d_{si}$, $i \in N$

Step 1:

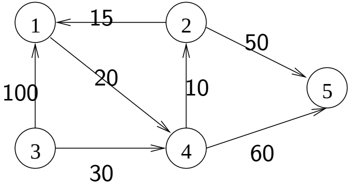
(a) If $\bar{S} = \emptyset$, stop. Otherwise, find node $j \in \bar{S}$ such that $y_j = \min_{i \in \bar{S}} \{y_i\}$. Set $S := S \cup \{j\}$ and $\bar{S} := \bar{S} \setminus \{j\}$

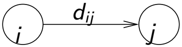
(b) For all $k \in \bar{S}$ and $i \in S$: If $y_k > y_i + d_{ik}$ set $y_k := y_i + d_{ik}$ and $pred(k) := i$. Repeat from (a).

- The vector *pred* keeps track of the predecessors
- Dijkstra's algorithm actually finds shortest paths from the source to *all* others nodes (this is not formulated in the LP)

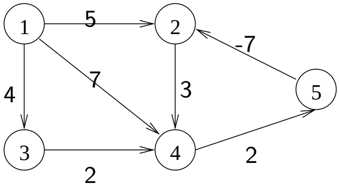
Example: Dijkstra's algorithm

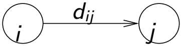
Find the shortest path from node 1 to all other nodes (Homework)





Negative arc lengths and negative cycles





- Negative length of arcs: extend Dijkstra's algorithm according to "move nodes back from S to $\bar{S}$" (known as Ford's algorithm)
- There may be a cycle of *negative* total length

$\Rightarrow$ "Length" of the shortest path $\rightarrow -\infty$

$\Rightarrow$ Ford's algorithm *either* finds a shortest path *or* detects a cycle with a negative total length

Algorithms for the shortest path problem: Floyd–Warshall (Ch. 8.4.2)

- Computes shortest paths between each pair of nodes
- Negative lengths are allowed; negative cycles will be detected
- Idea: Three nodes i, k, j and lengths d_{ik}, d_{kj}, and d_{ij}
- $i \rightarrow k \rightarrow j$ is a short-cut if $d_{ik} + d_{kj} < d_{ij}$
- In each iteration $1 \ldots k$, check whether d_{ij} can be improved by using the short-cut via k
- Administration of the algorithm: Maintain two matrices per iteration: $D[k]$ for the lengths and $pred[k]$ to keep track of the predecessor of each node

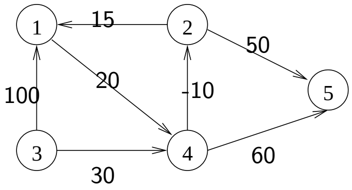
Floyd–Warshall's algorithm

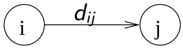
Floyd–Warshall's algorithm

Step 0: Initialize $D[0]$ and $pred[0]$

Step k:
- $D[k] := D[k-1]$, $pred[k] := pred[k-1]$
- For each element d_{ij} in $D[k]$:
  If $d_{ik} + d_{kj} < d_{ij}$, set $d_{ij} := d_{ik} + d_{kj}$ and $pred_{ij}[k] := k$
- Set $k := k + 1$
- If $k > n$ stop, else repeat Step k

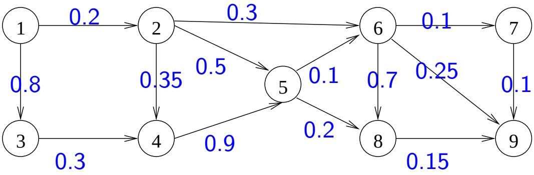
Find the shortest path from node 3 to all other nodes





Example: Most reliable route

- Mr Q drives to work daily
- Every link in the road network is patrolled by the police
- A probability $p_{ij} \in [0, 1]$ of *not* being stopped by the police is assigned to link (i, j)
- Mr Q wants to find the "shortest" (safest?) path in the sense that the probability of being stopped is as low as possible
- maximize $Prob$(not being stopped)



- Ex. $1 \to 4$: $\max\{p_{12}p_{24}; p_{13}p_{34}\} = \max\{0.2 \cdot 0.35; 0.8 \cdot 0.3\}$
- Note: This version *cannot* be formulated as a linear program

Alternative objectives $\Rightarrow$ Variants of Bellman's equations (Ch. 8.4.4)

Most reliable path (failure probability $p_{ij} \in [0, 1]$ for arc (i, j)):

- $y_s = 1$
- For all $j \in N \setminus \{s\}$:
  - $y_j = \max_{i \in N} \left\{ y_i \cdot p_{ij} : (i, j) \in A \right\}$

Cannot be formulated as a *linear* optimization problem

Highest capacity path (capacity $K_{ij} \geq 0$ on arc (i, j)):

- $y_s = \infty$
- For all $j \in N \setminus \{s\}$:
  - $y_j = \max_{i \in N} \left\{ \min\{y_i; K_{ij}\} : (i, j) \in A \right\}$

Can be formulated as a *linear* optimization problem. How?