MVE165/MMG631
Linear and integer optimization with applications
Lecture 12
Maximum flows and minimum cost flows—models and algorithms

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A district heating network

- Energy—in the form of hot water—is transported through a pipeline network with several sources and many destinations
- The network has several branches and junctions
- Pipe segment (i, j) has a maximum capacity of K_{ij} units of flow per time unit
- A pipe can be one- or bidirectional
- What is the maximum total amount of flow per time unit through this network?
- There may also be constraints on the temperature of the water at different points in the network
- Another application of the maximum flow model: evacuation of buildings (also time dynamics)

LP model for maximum flow problems

- Let x_{ij} denote the amount of flow through pipe segment (i, j) (flow direction $i \to j$)
- Let v denote the *total flow* from the source (node s) to the destination (node t)
- *Graph*: $G = (V, A, \mathbf{K})$ (nodes, directed arcs, arc capacities) (an undirected edge is represented by two directed arcs)

$$
\begin{array}{llr}
\max_{x,v} & v, & \text{maximize total flow, } t \text{ to } s \\
\text{s.t.} & \displaystyle\sum_{j:(s,j)\in A} (-x_{sj}) + v = 0, & \text{flow balance, node } s \\
& \displaystyle\sum_{i:(i,t)\in A} x_{it} - v = 0, & \text{flow balance, node } t \\
& \displaystyle\sum_{i:(i,k)\in A} x_{ik} + \sum_{j:(k,j)\in A} (-x_{kj}) = 0, \quad k \in V \setminus \{s, t\} & \text{flow balance, node } k \\
& x_{ij} \leq K_{ij}, \quad (i, j) \in A & \text{capacity, arc } (i, j) \\
& x_{ij} \geq 0, \quad (i, j) \in A & \text{nonnegative flow}
\end{array}
$$

[Draw!!]

A solution method for maximum flow problems (Edmonds & Karp, 1972)

1. Let $k := 0$, $v^0 := 0$, $x_{ij}^0 := 0$, and $u_{ij}^0 := K_{ij}$, $(i,j) \in A$.
2. Find a *maximum capacity path* $P^k \subset A$ from s to t (modified shortest path algorithm). The capacity of P^k is
   $\hat{u}^k := \min\left\{\min\left\{u_{ij}^k \,\middle|\, (i,j) \in P^k\right\}; \min\left\{x_{ij}^k \,\middle|\, (j,i) \in P^k\right\}\right\}$.
   If $\hat{u}^k = 0$, go to step 4.
3. Update the flows $x_{ij}^{k+1} := \begin{cases} x_{ij}^k + \hat{u}^k, & \text{if } (i,j) \in P^k, \\ x_{ij}^k - \hat{u}^k, & \text{if } (j,i) \in P^k, \\ x_{ij}^k, & \text{otherwise,} \end{cases}$
   the capacities $u_{ij}^{k+1} := \begin{cases} u_{ij}^k - \hat{u}^k, & \text{if } (i,j) \in P^k, \\ u_{ij}^k + \hat{u}^k, & \text{if } (j,i) \in P^k, \\ u_{ij}^k, & \text{otherwise,} \end{cases}$
   and the total flow $v^{k+1} := v^k + \hat{u}^k$. Let $k := k+1$, go to step 2.
4. The maximum total flow equals v^k.
   The flow solution is given by x_{ij}^k, $(i,j) \in A$.

LP dual of the maximum flow model

Primal

$$
\begin{array}{llll}
\max_{x,v} & & v, & \\
\text{s.t.} & -\displaystyle\sum_{j:(s,j)\in A} x_{sj} & +v = 0, & \\
& \displaystyle\sum_{i:(i,t)\in A} x_{it} & -v = 0, & \\
& \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = 0, & k \in V \setminus \{s,t\} \\
& 0 \leq x_{ij} & \leq K_{ij}, & (i,j) \in A
\end{array}
$$

Dual

$$
\begin{array}{llll}
\min_{\pi,\gamma} & \displaystyle\sum_{(i,j)\in A} K_{ij}\gamma_{ij}, & & \\
\text{s.t.} & -\pi_i + \pi_j \quad + \gamma_{ij} & \geq 0, & (i,j) \in A \\
& -\pi_t + \pi_s & = 1, & \\
& \pi_k & \text{free}, & k \in V, \\
& \gamma_{ij} & \geq 0, & (i,j) \in A
\end{array}
$$

[*Draw*!!]

Maximum flow – Minimum cut theorem

- An *(s,t)-cut* is a set of arcs which—when deleted—interrupt all flow in the network between the source s and the sink t
- The *cut capacity* equals the sum of capacities on all the arcs through the (s,t)-cut
- Finding the minimum (s,t)-cut is equivalent to solving the dual of the maximum flow problem

Theorem (Weak duality)

(i) *Each feasible flow x_{ij}, $(i,j) \in A$, yields a lower bound on v^*.*
(ii) *The capacity of each (s,t)-cut is an upper bound on v^*.*

Theorem (Strong duality)

value of maximum flow $=$ capacity of minimum cut

Optimal dual solution – minimum cut

Optimal values of the dual variables

$$\gamma_{ij} = \begin{cases} 1, & \text{if arc } (i,j) \text{ passes through the minimum cut,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\pi_k = \begin{cases} 1, & \text{if node } k \text{ can be reached from } s, \\ 0, & \text{otherwise.} \end{cases}$$

How is the minimum cut found using the Edmonds & Karp algorithm?

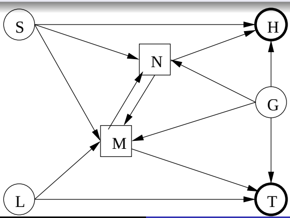
General minimum cost network flow problems

- A network consist of a set N of *nodes* linked by a set A of *arcs*
- A distance/cost c_{ij} is associated with each arc
- Each node i in the network has a net demand d_i
- Each arc carries an (unknown) amount of flow x_{ij} that is restricted by a maximum capacity $u_{ij} \in [0, \infty]$ and a minimum capacity $\ell_{ij} \in [0, u_{ij}]$
- The flow through each node must be *balanced*
- A network flow problem can be formulated as a linear program
- All extreme points of the feasible set are *integral* – due to the *unimodularity* property of the constraint matrix (see Ch. 8.6.3)

Minimum cost flow in a general network: Example

- Two paper mills: Holmsund and Tuna
- Three saw mills: Silje, Graninge and Lunden
- Two storage terminals: Norrstig and Mellansel

Facility	Supply (m^3)	Demand (m^3)
Silje	2400	
Graninge	1800	
Lunden	1400	
Holmsund		3500
Tuna		2100



Minimum cost flow in a general network: Example

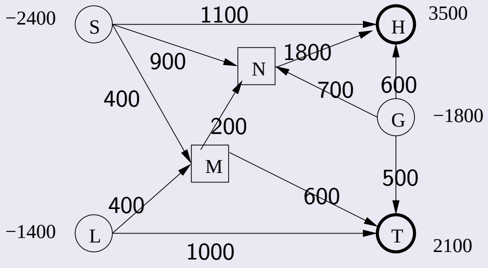
Transportation opportunities

From	To	Price/m^3	Capacity (m^3)
Silje	Norrstig	20	900
Silje	Mellansel	26	1000
Silje	Holmsund	45	1100
Graninge	Norrstig	8	700
Graninge	Mellansel	14	900
Graninge	Holmsund	37	600
Graninge	Tuna	22	600
Lunden	Mellansel	32	600
Lunden	Tuna	23	1000
Norrstig	Holmsund	11	1800
Norrstig	Mellansel	9	1800
Mellansel	Norrstig	9	1800
Mellansel	Tuna	9	1800

Minimum cost flow in a general network: Example

- Objective: Minimize transportation costs
- Satisfy demand
- Do not exceed the supply
- Do not exceed the transportation capacities

An optimal solution



Minimum cost flow in a general network: Example

$$
\begin{array}{lrcrl}
\min_x z := & 20x_{SN} + 26x_{SM} + 45x_{SH} + 8x_{GN} + 14x_{GM} & & & \\
& +37x_{GH} + 22x_{GT} + 32x_{LM} + 23x_{LT} + 11x_{NH} & & & \\
& +9x_{NM} + 9x_{MN} + 9x_{MT} & & & \\
\text{subject to} & -x_{SN} - x_{SM} - x_{SH} & = & -2400 & \text{(Silje)} \\
& -x_{GN} - x_{GM} - x_{GH} - x_{GT} & = & -1800 & \text{(Graninge)} \\
& -x_{LM} - x_{LT} & = & -1400 & \text{(Lunden)} \\
& x_{SN} + x_{GN} + x_{MN} - x_{NM} - x_{NH} & = & 0 & \text{(Norrstig)} \\
& x_{SM} + x_{LM} + x_{GM} + x_{NM} - x_{MN} - x_{MT} & = & 0 & \text{(Mellansel)} \\
& x_{SH} + x_{GH} + x_{NH} & = & 3500 & \text{(Holmsund)} \\
& x_{GT} + x_{LT} + x_{MT} & = & 2100 & \text{(Tuna)} \\
& 0 \leq x_{SN} & \leq & 900 & \\
& 0 \leq x_{SM} & \leq & 1000 & \\
& 0 \leq x_{SH} & \leq & 1100 & \\
& 0 \leq x_{GN} & \leq & 700 & \\
& 0 \leq x_{GM} & \leq & 900 & \\
& 0 \leq x_{GH} & \leq & 600 & \\
& 0 \leq x_{GT} & \leq & 600 & \\
& 0 \leq x_{LM} & \leq & 600 & \\
& 0 \leq x_{LT} & \leq & 1000 & \\
& 0 \leq x_{NH} & \leq & 1800 & \\
& 0 \leq x_{NM} & \leq & 1800 & \\
& 0 \leq x_{MN} & \leq & 1800 & \\
& 0 \leq x_{MT} & \leq & 1800 &
\end{array}
$$

The columns $\mathbf{A}_j$ of the equality constraint matrix ($\mathbf{Ax} = \mathbf{b}$) have one 1-element, one -1-element; the remaining elements are 0
$\Rightarrow$ the matrix $\mathbf{A}$ is totally unimodular

Minimum cost flows in general networks

- A network $G = (N, A)$ with nodes N and arcs A, $|N| = n$
- x_{ij} = flow through arc $(i, j) \in A$
- ℓ_{ij} and u_{ij} are lower and upper limits on x_{ij}
- c_{ij} = cost per unit flow on arc (i, j)
- d_i = demand in node i (negative demand = positive supply)

LP model

$$
\begin{array}{llll}
\min_x & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij}, & & \\
\text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = d_k, & & k \in N, \\
& \ell_{ij} \leq x_{ij} \leq u_{ij}, & & (i,j) \in A.
\end{array}
$$

Minimum cost flows in general networks: LP model and dual

The linear optimization model

$$
\begin{array}{lrcll}
\min_x & & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij}, & & \\
\text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & d_k, & k \in N, \\
& \ell_{ij} \leq x_{ij} & \leq & u_{ij}, & (i,j) \in A.
\end{array}
$$

Linear programming dual

$$
\begin{array}{lllll}
\max_{y,\alpha,\beta} & \displaystyle\sum_{k\in N} d_k y_k + \sum_{(i,j)\in A} \left(\ell_{ij}\alpha_{ij} - u_{ij}\beta_{ij} \right), & & & \\
\text{s.t.} & y_j - y_i & + \alpha_{ij} & - \beta_{ij} = c_{ij}, & (i,j) \in A, \\
& & \alpha_{ij} & , \ \beta_{ij} \geq 0, & (i,j) \in A.
\end{array}
$$

The simplex method for minimum cost network flows (Ch. 8.7)

A solution is optimal if

- the primal and dual solutions are feasible and
- the complementarity conditions are fulfilled

Reduced costs

$$\overline{c}_{ij} = c_{ij} + y_i - y_j, \qquad (i,j) \in A$$

Complementary conditions, $(i,j) \in A$

- $\alpha_{ij}(x_{ij} - \ell_{ij}) = 0$
- $\beta_{ij}(u_{ij} - x_{ij}) = 0$
- $x_{ij}(\overline{c}_{ij} - \alpha_{ij} + \beta_{ij}) = 0$

The simplex method for minimum cost network flows

Feasibility condition

Assume that $\ell_{ij} < u_{ij}$ holds for all $(i,j) \in A$

A feasible solution x_{ij}, $(i,j) \in A$, is optimal if the following hold

- $x_{ij} = u_{ij} \Rightarrow \alpha_{ij} = 0$ $\quad\Rightarrow$ Reduced cost: $\overline{c}_{ij} = -\beta_{ij} \leq 0$
- $x_{ij} = \ell_{ij} \Rightarrow \beta_{ij} = 0$ $\quad\Rightarrow$ Reduced cost: $\overline{c}_{ij} = \alpha_{ij} \geq 0$
- $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow \alpha_{ij} = \beta_{ij} = 0$ $\quad\Rightarrow$ Reduced cost: $\overline{c}_{ij} = 0$

A *basic solution* is characterized by the following

- If $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow$ the arc (i,j) is in the *basis*
  $\Leftrightarrow x_{ij}$ is a basic variable
- If $x_{ij} = \ell_{ij}$ or $x_{ij} = u_{ij} \Rightarrow$ the arc (i,j) *may* be in the *basis*
  $\Leftrightarrow x_{ij}$ *may be* a basic variable
- The $n-1$ *basic arcs* form a *spanning tree* in G (one primal equation is a linear combination of the rest – can be removed)

The simplex method for minimum cost flows

1. Find a feasible solution (a spanning tree of basic arcs)[a]
2. Compute reduced costs $\overline{c}_{ij} = c_{ij} + y_i - y_j$ for all non-basic arcs
3. Check termination criteria: If, for every arc (i,j),
   - either: $\overline{c}_{ij} = 0$ and $\ell_{ij} \leq x_{ij} \leq u_{ij}$,
   - or: $\overline{c}_{ij} < 0$ and $x_{ij} = u_{ij}$,
   - or: $\overline{c}_{ij} > 0$ and $x_{ij} = \ell_{ij}$

   hold, then STOP. x_{ij}, $(i,j) \in A$ form an optimal solution
4. *Entering variable (arc)*: $(p,q) \in \arg\max_{(i,j)\in I} |\overline{c}_{ij}|$
   $I =$ the set of non-basic arcs *not* fulfilling the conditions in 3
5. *Leaving variable (arc)*: Send flow along the cycle defined by the current *basis* (spanning tree) and the arc (p,q). The arc (i,j) whose flow x_{ij} first reaches u_{ij} or ℓ_{ij} leaves the basis
6. Go to step 2

[a] For the basic arcs (variables), the reduced costs $\bar{c}_{ij} := c_{ij} + y_i - y_j = 0$. Letting $y_1 := 0$ the values of y_i, $i \in N$, are then given by these equalities.