MVE165/MMG631
Linear and integer optimization with applications
Lecture 5
Linear programming duality

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An "intuitive" derivation of duality, I (Ch. 6.1)

A linear program with optimal value z^*

$$
\begin{array}{rlrrlrc}
z^* = \max & z := & 20x_1 & +18x_2 & & & \text{weights} \\
& \text{subject to} & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
& & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
& & & x_1, x_2 & \geq 0 & &
\end{array}
$$

[DRAW GRAPH]

- What is the largest possible value of z (i.e., z^*)?

Compute upper estimates of z^*, e.g.:

- Multiply (1) by 3:

$$\Rightarrow \quad 21x_1 + 30x_2 \leq 10800 \qquad \Rightarrow \quad z^* \leq 10800$$

- Multiply (2) by 1.5:

$$\Rightarrow \quad 24x_1 + 18x_2 \leq 8100 \qquad \Rightarrow \quad z^* \leq 8100$$

- Combine: $0.6\times(1)+1\times(2)$:

$$\Rightarrow \quad 20.2x_1 + 18x_2 \leq 7560 \qquad \Rightarrow \quad z^* \leq 7560$$

An "intuitive" derivation of duality, II (Ch. 6.1)

A linear program with optimal value z^*

$$
\begin{array}{llrrlcc}
\text{maximize} & z := & 20x_1 & +18x_2 & & & \text{weights} \\
\text{subject to} & & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
 & & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
 & & & x_1, x_2 & \geq 0 & &
\end{array}
$$

[Draw graph]

- Do better than guess—compute *optimal* weights!
- Value of estimate: $w = 3600v_1 + 5400v_2 \to \min$

Constraints on the weights

$$
\begin{array}{rl}
7v_1 + 16v_2 & \geq 20 \\
10v_1 + 12v_2 & \geq 18 \\
v_1, v_2 & \geq 0
\end{array}
$$

An "intuitive" derivation of duality, III (Ch. 6.1)

The best (lowest) possible upper estimate of z^*

$$\begin{array}{rrrrrl}
\text{minimize} & w := & 3600v_1 & + & 5400v_2 & \\
\text{subject to} & & 7v_1 & + & 16v_2 & \geq 20 \\
& & 10v_1 & + & 12v_2 & \geq 18 \\
& & & & v_1, v_2 & \geq 0
\end{array}$$

- A linear program! [DRAW GRAPH!!]
- It is called the *linear programming dual* of the original linear program

The lego model – the market problem

Consider the lego problem

$$\begin{array}{lrcrcrcl} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

- Option: Sell bricks instead of making furniture
- $v_1(v_2) =$ price of a large (small) brick
- The market wishes to *minimize the payment:* $\min\ 6v_1 + 8v_2$

Sell only if prices are high enough

- $2v_1 + 2v_2 \geq 1600$ – otherwise better to make tables
- $v_1 + 2v_2 \geq 1000$ – otherwise better to make chairs
- $v_1, v_2 \geq 0$ – don't sell at a negative price

A general linear program on "standard form"

A linear program with n non-negative variables, m equality constraints ($m < n$), and non-negative right-hand-sides

$$
\begin{array}{lrcll}
\text{maximize} & z = \displaystyle\sum_{j=1}^{n} c_j x_j, & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & = & b_i, & i = 1, \ldots, m, \\
 & x_j & \geq & 0, & j = 1, \ldots, n,
\end{array}
$$

where

$x_j \in \mathbb{R}, \quad j = 1, \ldots, n,$
$a_{ij} \in \mathbb{R}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$
$b_i \geq 0, \quad i = 1, \ldots, m,$
$c_j \in \mathbb{R}, \quad j = 1, \ldots, n.$

Or, on matrix form

$$
\begin{array}{lr}
\text{maximize} & z = \mathbf{c}^{\mathrm{T}} \mathbf{x}, \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \geq \mathbf{0}^n,
\end{array}
$$

where

$\mathbf{x} \in \mathbb{R}^n,$
$\mathbf{A} \in \mathbb{R}^{m \times n},$
$\mathbf{b} \in \mathbb{R}_+^m,$
$\mathbf{c} \in \mathbb{R}^n.$

Linear programming duality

To each primal linear program corresponds a dual linear program

(Primal)

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}^n \end{aligned}$$

(Dual)

$$\begin{aligned} \text{maximize} \quad & w = \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c} \end{aligned}$$

The component forms of the primal and dual programs

(Primal)

$$\begin{aligned} \min \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

(Dual)

$$\begin{aligned} \max \quad & w = \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \\ & j = 1, \ldots, n \end{aligned}$$

An example of linear programming duality

A primal linear program

$$\begin{array}{lrrrll}
\text{minimize} & z = & 2x_1 & +3x_2 & & & \text{weights/duals} \\
\text{subject to} & & 3x_1 & +2x_2 & = & 14 & y_1 \\
 & & 2x_1 & -4x_2 & \geq & 2 & y_2 \\
 & & 4x_1 & +3x_2 & \leq & 19 & y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

The corresponding dual linear program

$$\begin{array}{lrrrrll}
\text{maximize} & w = & 14y_1 & +2y_2 & +19y_3 & & & \text{weights/primals} \\
\text{subject to} & & 3y_1 & +2y_2 & +4y_3 & \leq & 2 & x_1 \\
 & & 2y_1 & -4y_2 & +3y_3 & \leq & 3 & x_2 \\
 & & y_1 & & & & \text{free} & \\
 & & & y_2 & & \geq & 0 & \\
 & & & & y_3 & \leq & 0 &
\end{array}$$

Rules for constructing the dual program (Ch. 6.2)

maximization $\Longleftrightarrow$ minimization

dual program	$\Longleftrightarrow$	primal program
primal program	$\Longleftrightarrow$	dual program
constraints	$\Longleftrightarrow$	*variables*
$\geq$	$\Longleftrightarrow$	≤ 0
$\leq$	$\Longleftrightarrow$	≥ 0
$=$	$\Longleftrightarrow$	free
variables	$\Longleftrightarrow$	*constraints*
≥ 0	$\Longleftrightarrow$	$\geq$
≤ 0	$\Longleftrightarrow$	$\leq$
free	$\Longleftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

Primal			Dual		
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	(1a)	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$	(2a)
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	(1b)	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$	(2b)
	$\mathbf{x} \geq \mathbf{0}^n$	(1c)			

Weak duality [Th. 6.1]

Let $\mathbf{x}$ be a feasible point in the primal (minimization) and $\mathbf{y}$ be a feasible point in the dual (maximization). Then, it holds that

$$z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y} = w$$

Proof: $z \underbrace{=}_{(1a)} \mathbf{c}^{\mathrm{T}}\mathbf{x} \underbrace{\geq}_{(2b),\ (1c)} \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x} \underbrace{=}_{(1b)} \mathbf{y}^{\mathrm{T}}\mathbf{b} \underbrace{=}_{(2a)} w.$ $\square$

In the course book, the primal is formulated with inequality constraints in (1b): adjust the dual and the proof for that case!

Primal		Dual	
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \le \mathbf{c}$
	$\mathbf{x} \ge \mathbf{0}^n$		

Corollary [Th. 6.2]

If $\bar{\mathbf{x}}$ is feasible in the primal and $\bar{\mathbf{y}}$ is feasible in the dual, and it holds that

$$\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} = \mathbf{b}^{\mathrm{T}}\bar{\mathbf{y}},$$

then $\bar{\mathbf{x}}$ is optimal in the primal and $\bar{\mathbf{y}}$ is optimal in the dual.

Duality properties, III (Ch. 6.3)

Primal		Dual	
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}^n$		

Strong duality [Th. 6.3]

In a pair of primal and dual linear programs, if one of them has a bounded optimal solution $\hat{\mathbf{x}}$ (or $\hat{\mathbf{y}}$), so does the other, i.e., $\hat{\mathbf{y}}$ (or $\hat{\mathbf{x}}$), and their optimal values are equal, i.e. $\mathbf{c}^{\mathrm{T}}\hat{\mathbf{x}} = \mathbf{b}^{\mathrm{T}}\hat{\mathbf{y}}$.

Duality properties, IV (Ch. 6.3)

Primal		Dual	
minimize	$z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$	maximize	$w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} = \mathbf{b}$	subject to	$\mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}^n$		

Complementary slackness [Th. 6.5; proof in the course book]

If $\mathbf{x}$ is *optimal in the primal* and $\mathbf{y}$ is *optimal in the dual*, then it holds that

$$\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0.$$

If $\mathbf{x}$ is *feasible in the primal*, $\mathbf{y}$ is *feasible in the dual*, and $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$, then

$\mathbf{x}$ and $\mathbf{y}$ are optimal in their respective problems.

Primal

$$z^* := \min z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N$$
$$\text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$
$$\mathbf{x}_B \geq \mathbf{0}^m, \mathbf{x}_N \geq \mathbf{0}^{n-m}$$

Dual

$$w^* := \max w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$$
$$\text{subject to} \quad \mathbf{B}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}_B$$
$$\mathbf{N}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}_N$$

Duality theorem [Th. 6.4]

Assume that $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ is an optimal basic (feasible) solution to the primal problem. Then $\mathbf{y}^{\mathrm{T}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ is an optimal solution to the dual problem and $z^* = w^*$.

Proof structure [full proof in the course book]

1. $\mathbf{y}^{\mathrm{T}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ is feasible in the dual problem
2. The optimal objective values z^* and w^* are equal
3. Follows from complementarity: $(\mathbf{x}_B, \mathbf{x}_N)$ and $\mathbf{y}$ are feasible in the primal and dual respective problem and $z^* = w^*$

Relations between primal and dual optimal solutions

primal (dual) problem	$\Longleftrightarrow$	dual (primal) problem
unique and non-degenerate solution	$\Longleftrightarrow$	unique and non-degenerate solution
unbounded solution	$\Longrightarrow$	no feasible solutions
no feasible solutions	$\Longrightarrow$	unbounded solution *or* no feasible solutions
degenerate solution	$\Longleftrightarrow$	alternative solutions

Homework

Exercises on linear programming duality

- Formulate and solve graphically the dual of:

$$\begin{array}{lrrrrl} \text{minimize} & z = & 6x_1 & +3x_2 & +x_3 & \\ \text{subject to} & & 6x_1 & -3x_2 & +x_3 & \geq 2 \\ & & 3x_1 & +4x_2 & +x_3 & \geq 5 \\ & & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

- Then find the optimal primal solution
- Verify that the dual of the dual equals the primal