MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 3
Extreme points of convex polyhedra; reformulations; basic feasible solutions

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Course evaluation

- The first meeting will be held on Friday, March 29 at 9:30
- The second meeting will be held during week 16 or 18
- Notes will be published in the course's PingPong event
- Any voluntary representative from GU is also welcome! Anyone?
- Student representatives to be decided. Contact these to present your opinion

Linear optimization models = linear programs (LP)

A linear optimization model: c_j, a_{ij}, b_i: constant parameters

$$\begin{aligned} \textit{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \textit{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

In vector/matrix notation: $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$\begin{aligned} \textit{min} \quad & z = \mathbf{c}^\top \mathbf{x} \\ \textit{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}^n \end{aligned}$$

A polyhedron

The *feasible region* of a linear optimization model is defined as the intersection of halfspaces in $\mathbb{R}^n$ defined by its constraints.

The feasible region is a polyhedron, denoted $X \subset \mathbb{R}^n_+$

$$X := \left\{ \mathbf{x} \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, \dots, m \right\} \equiv \left\{ \mathbf{x} \geq \mathbf{0}^n \;\middle|\; \mathbf{Ax} \leq \mathbf{b} \right\}$$

[Draw on the board]

Convex combinations (Ch. 4.1)

Definition (Convex combination (Def. 4.1))

A *convex combination* of the points $\mathbf{x}^p \in \mathbb{R}^n$, $p = 1, \ldots, P$, is any point $\mathbf{x} \in \mathbb{R}^n$ that can be expressed as

$$\mathbf{x} = \sum_{p=1}^{P} \lambda_p \mathbf{x}^p$$

where the following constraints hold:

$$\sum_{p=1}^{P} \lambda_p = 1; \qquad \lambda_p \geq 0, \quad p = 1, \ldots, P$$

The variables λ_p are called *convexity weights*

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Convex sets (Ch. 2.4)

Definition (Convex set (Def. 2.5))

A set $X \in \mathbb{R}^n$ is a *convex set* if, for any two points $\mathbf{x}^1 \in X$ and $\mathbf{x}^2 \in X$, and any $\lambda \in [0, 1]$, it holds that

$$\mathbf{x} := \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2 \in X$$

[Draw on the board]

Linear programs and convex polyhedra (Ch. 4.1)

Let $\mathbf{x} := \lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$, where $\mathbf{x}^1$ and $\mathbf{x}^2$ are feasible, i.e.,

$\mathbf{x}^1 \geq \mathbf{0}^n$, $\mathbf{x}^2 \geq \mathbf{0}^n$, $\mathbf{A}\mathbf{x}^1 \leq \mathbf{b}$, and $\mathbf{A}\mathbf{x}^2 \leq \mathbf{b}$ hold.

The intersection of linearly constrained regions forms a convex set

The feasible region of a linear program is a *convex set*, since for any two feasible points $\mathbf{x}^1$ and $\mathbf{x}^2$ and any $\lambda \in [0,1]$ it holds that

$$\begin{aligned}
\sum_{j=1}^n a_{ij}x_j &= \sum_{j=1}^n a_{ij}\left(\lambda x_j^1 + (1-\lambda)x_j^2\right) \\
&= \lambda \sum_{j=1}^n a_{ij}x_j^1 + (1-\lambda)\sum_{j=1}^n a_{ij}x_j^2 \\
&\leq \lambda b_i + (1-\lambda)b_i \\
&= b_i, \qquad\qquad i = 1,\ldots,m
\end{aligned}$$

and

$$x_j = \lambda x_j^1 + (1-\lambda)x_j^2 \geq 0, \qquad\qquad j = 1,\ldots,n$$

[Draw on the board]

Extreme points (Ch. 4.1)

Definition (Extreme point (Def. 4.2))

The point $\mathbf{x}^k$ is an *extreme point* of the polyhedron X if $\mathbf{x}^k \in X$ and it is *not* possible to express $\mathbf{x}^k$ as a *strict convex combination* of two distinct points in X.

I.e: Given $\mathbf{x}^1 \in X$, $\mathbf{x}^2 \in X$, and $0 < \lambda < 1$, it holds that $\mathbf{x}^k = \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2$ only if $\mathbf{x}^k = \mathbf{x}^1 = \mathbf{x}^2$ hold.

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Optimal solution and extreme points (Ch. 4.1)

Theorem (Optimal solution in an extreme point (Th. 4.2))

Assume that the feasible region $X = \{\mathbf{x} \geq \mathbf{0}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ *is non-empty and bounded.*
Then, the minimum value of the objective $\mathbf{c}^\top \mathbf{x}$ *is attained at (at least) one extreme point* $\mathbf{x}^k$ *of* X.

[Draw on the board]

Proof of Theorem 4.2

Assume the opposite, that there is a *non-extreme point* $\tilde{\mathbf{x}} \in X$ with a lower objective value than any of the *extreme points*, i.e.,

$$\mathbf{c}^\top \tilde{\mathbf{x}} < \mathbf{c}^\top \mathbf{x}^k \quad \text{for all extreme points} \quad \mathbf{x}^k \text{ of } X. \tag{1}$$

Since the polyhedron X is a convex set, the point $\tilde{\mathbf{x}}$ can be expressed as a convex combination of the extreme points of X, i.e.,

$$\begin{aligned} \tilde{\mathbf{x}} &= \textstyle\sum_{k=1}^{p} \lambda_k \mathbf{x}^k; && (2) \\ \textstyle\sum_{k=1}^{p} \lambda_k &= 1 && (3) \\ \lambda_k &\geq 0, \qquad k = 1, \ldots, p && (4) \end{aligned}$$

where p is the number of extreme points. Then, it must hold that

$$\mathbf{c}^\top \tilde{\mathbf{x}} \underbrace{=}_{(2)} \mathbf{c}^\top \sum_{k=1}^{p} \lambda_k \mathbf{x}^k = \sum_{k=1}^{p} \lambda_k \mathbf{c}^\top \mathbf{x}^k \underbrace{>}_{(1),(4)} \sum_{k=1}^{p} \lambda_k \mathbf{c}^\top \tilde{\mathbf{x}} = \mathbf{c}^\top \tilde{\mathbf{x}} \sum_{k=1}^{p} \lambda_k \underbrace{=}_{(3)} \mathbf{c}^\top \tilde{\mathbf{x}}$$

which is a contradiction.

A general linear program — notation

Definition (Notation of linear programs)

minimize or maximize $\quad c_1 x_1 + \ldots + c_n x_n$

$$\text{subject to} \quad a_{i1} x_1 + \ldots + a_{in} x_n \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \ldots, m$$

$$x_j \left\{ \begin{array}{l} \leq 0 \\ \text{unrestricted in sign} \\ \geq 0 \end{array} \right\}, \quad j = 1, \ldots, n$$

The blue notation corresponds to the *standard form*

The standard form of the simplex method for linear programs (Ch. 4.2)

- Every linear program can be reformulated such that:
  - all constraints are expressed as *equalities* with *non-negative right hand sides*
  - all variables involved are restricted to be *non-negative*
- Referred to as the *standard form*
- These requirements streamline the calculations of the *simplex method*
- Software solvers (e.g., Clp, Gurobi, Cplex, GLPK, SCIP) handle also inequality constraints and unrestricted variables — the reformulations are made automatically

The simplex method — standard form reformulations

- Slack variables, s_i:

$$\left[\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i, & \forall i \\ x_j & \geq & 0, & \forall j \end{array}\right] \iff \left[\begin{array}{rrcll} \sum_{j=1}^{n} a_{ij}x_j & +s_i & = & b_i, & \forall i \\ x_j & & \geq & 0, & \forall j \\ & s_i & \geq & 0, & \forall i \end{array}\right]$$

- The lego example:

$$\left[\begin{array}{rrcr} 2x_1 & +x_2 & \leq & 6 \\ 2x_1 & +2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{array}\right] \iff \left[\begin{array}{rrrrcr} 2x_1 & +x_2 & +s_1 & & = & 6 \\ 2x_1 & +2x_2 & & +s_2 & = & 8 \\ & & & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}\right]$$

- s_1 and s_2 are called *slack variables*—they "fill out" the (positive) distances between the left and right hand sides

The simplex method — standard form reformulations

- Surplus variables, s_i:

$$\left[\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \geq & b_i, & \forall i \\ x_j & \geq & 0, & \forall j \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrcll} \sum_{j=1}^{n} a_{ij}x_j & -s_i & = & b_i, & \forall i \\ x_j & & \geq & 0, & \forall j \\ & s_i & \geq & 0, & \forall i \end{array}\right]$$

- Surplus variable s_3 (another instance):

$$\left[\begin{array}{rcrcr} x_1 & + & x_2 & \geq & 800 \\ & & x_1, x_2 & \geq & 0 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rcrcrcr} x_1 & + & x_2 & - & s_3 & = & 800 \\ & & & & x_1, x_2, s_3 & \geq & 0 \end{array}\right]$$

The simplex method — standard form reformulations

- Suppose that $b < 0$:

$$\left[\begin{array}{r} \sum_{j=1}^{n} a_j x_j \leq b \\ x_j \geq 0, \forall j \end{array}\right] \Longleftrightarrow \left[\begin{array}{r} \sum_{j=1}^{n} (-a_j) x_j \geq -b \\ x_j \geq 0, \forall j \end{array}\right] \Longleftrightarrow \left[\begin{array}{rrl} -\sum_{j=1}^{n} a_j x_j & -s & = -b \\ x_j & & \geq 0, \forall j \\ & s & \geq 0 \end{array}\right]$$

- Non-negative right hand side:

$$\left[\begin{array}{rl} x_1 - x_2 & \leq -23 \\ x_1, x_2 & \geq 0 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rl} -x_1 + x_2 & \geq 23 \\ x_1, x_2 & \geq 0 \end{array}\right] \Longleftrightarrow \left[\begin{array}{rl} -x_1 + x_2 - s_4 & = 23 \\ x_1, x_2, s_4 & \geq 0 \end{array}\right]$$

The simplex method — standard form reformulations

- Suppose that some of the variables are unconstrained (here: $k < n$). Replace x_j with $x_j^1 - x_j^2$ for the corresponding indices:

$$\begin{bmatrix} \sum_{j=1}^{n} a_j x_j \leq b \\ x_j \geq 0,\ j = 1, \dots, k \end{bmatrix} \iff \begin{bmatrix} \sum_{j=1}^{k} a_j x_j + \sum_{j=k+1}^{n} a_j (x_j^1 - x_j^2) + s & = b & \\ x_j \geq 0, & j = 1, \dots, k, \\ x_j^1 \geq 0, x_j^2 \geq 0, & j = k+1, \dots, n \\ s \geq 0 & \end{bmatrix}$$

- Sign-restricted (non-negative) variables:

$$\begin{bmatrix} x_1 + x_2 \leq 10 \\ x_1 \geq 0 \end{bmatrix} \iff \begin{bmatrix} x_1 + x_2^1 - x_2^2 \leq 10 \\ x_1, x_2^1, x_2^2 \geq 0 \end{bmatrix} \iff \begin{bmatrix} x_1 + x_2^1 - x_2^2 + s_5 = 10 \\ x_1, x_2^1, x_2^2, s_5 \geq 0 \end{bmatrix}$$

Basic solutions (Ch. 4.3)

- Consider m equations and n variables, where $m \leq n$
- Set $n - m$ variables to zero and solve (if possible) the remaining $m \times m$ system of equations
- If the solution is *unique*, it is called a *basic* solution

Definition (Def. 4.3)

A *basic* solution to the $m \times n$ system of equations $\mathbf{Ax} = \mathbf{b}$ is obtained if $n - m$ of the variables are set to 0 and the remaining variables get their unique values from the solution to the remaining $m \times m$ system of equations.

The variables that are set to 0 are called *nonbasic variables* and the remaining m variables are called *basic variables*.

Basic feasible solutions (BFS) (Ch. 4.3)

- A basic solution $\mathbf{x}$ corresponds to the *intersection* of m hyperplanes in $\mathbb{R}^m$
  - It is *feasible* if $\mathbf{x} \geq \mathbf{0}$
  - It is *infeasible* if $\mathbf{x} \ngeq \mathbf{0}$
- Each *extreme point* of the feasible set is an intersection of m hyperplanes such that all variable values are ≥ 0
- Basic feasible solution $\Longleftrightarrow$ extreme point of the feasible set

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 & \qquad x_1 &\geq 0 \\
a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 & \qquad x_2 &\geq 0 \\
\ldots & & \ldots & \\
a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m & \qquad x_n &\geq 0
\end{aligned}
$$

Basic feasible solutions – algebraic descriptions

Assume that $m < n$ and that $b_i \geq 0$, $i = 1, \ldots, m$, and let

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Consider the linear program to

$$\begin{aligned} \underset{\mathbf{x}}{\textit{minimize}} \quad & z = \mathbf{c}^\top \mathbf{x} \\ \textit{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Partition $\mathbf{x}$ into m basic variables $\mathbf{x}_B$ and $n - m$ non-basic variables $\mathbf{x}_N$, such that $\mathbf{x}^\top = (\mathbf{x}_B^\top, \mathbf{x}_N^\top)$
- Analogously, let $\mathbf{c}^\top = (\mathbf{c}_B^\top, \mathbf{c}_N^\top)$ and $\mathbf{A} = (\mathbf{A}_B, \mathbf{A}_N) \equiv (\mathbf{B}, \mathbf{N})$
- The matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$ with inverse $\mathbf{B}^{-1}$ (if it exists)

Basic feasible solutions – algebraic descriptions (Ch. 4.8)

Rewrite the linear program equivalently as

$$\begin{aligned}
\textit{minimize} \quad & z = \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N & \text{(5a)}\\
\textit{subject to} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} & \text{(5b)}\\
& \mathbf{x}_B \geq \mathbf{0}^m, \ \mathbf{x}_N \geq \mathbf{0}^{n-m} & \text{(5c)}
\end{aligned}$$

- Multiply the system of equations (5b) by $\mathbf{B}^{-1}$ from the left:

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

$$\Longrightarrow \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) \qquad (6)$$

- Replace $\mathbf{x}_B$ in (5a) by the expression in (6):

$$\mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N = \mathbf{c}_B^\top \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^\top \mathbf{x}_N = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N$$

$$\begin{aligned}
\Rightarrow \quad \textit{minimize} \quad & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N \\
\textit{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \ \mathbf{x}_N \geq \mathbf{0}^{n-m}
\end{aligned}$$

Basic feasible solutions – algebraic descriptions

The rewritten program

$$\begin{aligned} \textit{minimize} \quad & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N && \text{(7a)} \\ \textit{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m && \text{(7b)} \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} && \text{(7c)} \end{aligned}$$

At the basic solution defined by $B \subset \{1, \ldots, n\}$:

- Each non-basic variable takes the value 0, i.e., $\mathbf{x}_N = \mathbf{0}$
- The basic variables take the values $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$
- The value of the objective function is $z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$
- The basic solution is feasible if $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}^m$

Basic feasible solutions, example

- Constraints:

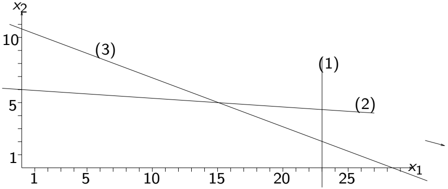
$$\begin{array}{rcrcll} x_1 & & & \leq & 23 & (1) \\ 0.067x_1 & + & x_2 & \leq & 6 & (2) \\ 3x_1 & + & 8x_2 & \leq & 85 & (3) \\ & & x_1, x_2 & \geq & 0 & \end{array}$$

- Add slack variables:

$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \quad (1) \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \quad (2) \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \quad (3) \\ & & & & x_1, x_2, s_1, s_2, s_3 & \geq 0 \end{array}$$

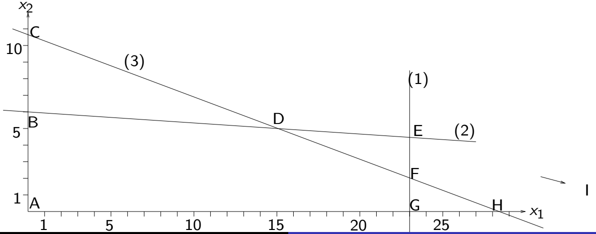
$m = 3$

$n = 5$

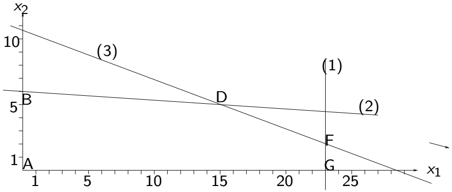


Basic and non-basic variables and solutions

basic variables	basic solution			non-basic variables $(0, 0)$	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic *feasible* solutions correspond to solutions to the system of equations that *fulfill non-negativity*



$$\begin{array}{rrrrl} x_1 & & +s_1 & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 \quad & = 6 \\ 3x_1 & +8x_2 & & \quad +s_3 & = 85 \end{array}$$

A: $x_1 = x_2 = 0 \Rightarrow \left[\begin{array}{cccl} s_1 & & & = 23 \\ & s_2 & & = 6 \\ & & s_3 & = 85 \end{array}\right]$

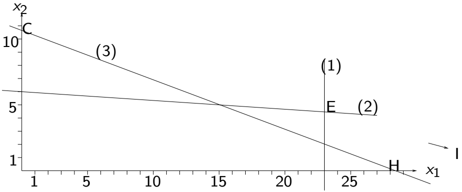
B: $x_1 = s_2 = 0 \Rightarrow \left[\begin{array}{cccl} & s_1 & & = 23 \\ x_2 & & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$

D: $s_3 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & +s_1 & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

F: $s_3 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

G: $x_2 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +s_2 & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

Basic *infeasible* solutions corresp. to solutions to the system of equations with one or more variables < 0



$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}$$

H: $x_2 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & +s_2 & = 6 \\ 3x_1 & & & = 85 \end{array}\right]$

C: $x_1 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} & s_1 & & = 23 \\ x_2 & & +s_2 & = 6 \\ 8x_2 & & & = 85 \end{array}\right]$

I: $s_2 = x_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

-: $s_1 = x_1 = 0 \Rightarrow \left[\begin{array}{rrrrl} & & & 0 & = 23 \\ x_2 & +s_2 & & & = 6 \\ 8x_2 & & +s_3 & & = 85 \end{array}\right]$

E: $s_1 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & +s_3 & = 85 \end{array}\right]$