MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 4
Linear programming: the simplex algorithm; degeneracy; unbounded solution; multiple optimal solutions; infeasibility; starting solutions

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2019–04–02

Basic feasible solutions – algebraic descriptions (Ch. 4.8)

A linear program with basis B (the square matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$ is nonsingular):

$$\begin{aligned} \textit{minimize} \quad & z = \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N && \text{(1a)} \\ \textit{subject to} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} && \text{(1b)} \\ & \mathbf{x}_B \geq \mathbf{0}^m, \ \mathbf{x}_N \geq \mathbf{0}^{n-m} && \text{(1c)} \end{aligned}$$

Multiply the system of equations (1b) by $\mathbf{B}^{-1}$ from the left $\Longrightarrow$ an equivalent formulation:

$$\begin{aligned} \textit{minimize} \quad & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N && \text{(2a)} \\ \textit{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m && \text{(2b)} \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} && \text{(2c)} \end{aligned}$$

Basic feasible solutions – algebraic descriptions (Ch. 4.8)

The equivalent formulation:

$$\begin{array}{lrcl} \textit{minimize} & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N & & \\ \textit{subject to} & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N & \geq & \mathbf{0}^m \\ & \mathbf{x}_N & \geq & \mathbf{0}^{n-m} \end{array}$$

At the basic solution defined by $B \subset \{1, \dots, n\}$:

- Each non-basic variable takes the value 0, i.e., $\mathbf{x}_N = \mathbf{0}^{n-m}$
- The basic variables take the values $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$
- The value of the objective function is $z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$
- The basic solution is feasible if $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}^m$

Optimality in an extreme point

Recall Theorem 4.2:

Theorem (optimal solution in an extreme point)

Assume that the feasible region $X = \{\mathbf{x} \geq \mathbf{0}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ *is non-empty and bounded.*
Then, the minimum value of the objective $\mathbf{c}^\top \mathbf{x}$ *is attained at (at least) one extreme point* $\bar{\mathbf{x}}$ *of* X*.*

Also (Ch. 4.3)

Each extreme point $\bar{\mathbf{x}}$ of X corresponds to a *basic feasible solution* (BFS), i.e.,
an $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B, \bar{\mathbf{x}}_N) \in X$ such that $\bar{\mathbf{x}}_N = \mathbf{0}^{n-m}$ and $\bar{\mathbf{x}}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}^m$

$\Longrightarrow$

Search for BFSs, with iteratively better (lower) objective values

The objective value of a BFS

The equivalent formulation:

$$\begin{array}{lrcl} \textit{minimize} & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N & & \\ \textit{subject to} & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N & \geq & \mathbf{0}^m \\ & \mathbf{x}_N & \geq & \mathbf{0}^{n-m} \end{array}$$

- The objective value for the basis B is $z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$
- If, for some (non-basic) $j \in N$, it holds that $c_j - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{A}_j < 0$, then the objective value z will decrease when the value of the variable x_j increases from 0
- To stay feasible, the value of the non-basic variable x_j may increase until $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j = 0$ for some $i \in B$

The simplex method: Optimality and feasibility and change of basis (Ch. 4.4)

Optimality condition (for minimization)

The basis B is optimal if $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
(marginal values = reduced costs ≥ 0)

If not, choose as entering variable $j \in N$ the one with the lowest (negative) value of the reduced cost $c_j - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{A}_j$

Feasibility condition

For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

Choose the leaving variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

Simplex search for linear optimization (Ch. 4.6)

Overview of the simplex algorithm for linear optimization (minimization)

1. **Initialization:** Choose any *feasible basis*, construct the corresponding *basic solution* $\mathbf{x}^0$, let $t := 0$
2. **Step direction:** Select a variable to *enter the basis* using the *optimality condition* (negative marginal value). Stop if no entering variable exists
3. **Step length:** Use the *feasibility condition* (smallest non-negative quotient) to select a variable to *leave the basis*
4. **New iterate:** Compute the *new basic solution* $\mathbf{x}^{t+1}$ by performing matrix operations
5. Let $t := t + 1$ and repeat from step 2

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n - m$ *non-basic* variables

Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1, s_2, s_3$ are *basic*

$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}$$

Express s_1, s_2, and s_3 in terms of x_1 and x_2 (*non-basic*):

$$\begin{array}{rcrrr} s_1 & = & 23 & -x_1 & \\ s_2 & = & 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 & = & 85 & -3x_1 & -8x_2 \end{array}$$

- We wish to maximize the value of the objective function $2x_1 + 3x_2$

Express the objective in terms of the *non-basic* variables:

$$\text{(maximize)} \quad z = 2x_1 + 3x_2 \quad \Leftrightarrow \quad z - 2x_1 - 3x_2 = 0$$

Basic feasible solutions and the simplex method

The *first basic solution* can be represented as

$$\begin{array}{lcccccl|c} z & -2x_1 & -3x_2 & & & & = 0 & (0) \\ & x_1 & & +s_1 & & & = 23 & (1) \\ & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\ & 3x_1 & +8x_2 & & & +s_3 & = 85 & (3) \end{array}$$

- Marginal values for increasing the non-basic variables x_1 and x_2 from zero: 2 and 3, resp.

$\Rightarrow$ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1, s_2, or s_3) must *leave the basis*. Which?

The value of x_2 increases until a basic variable reaches the value 0:

$$\left.\begin{array}{ll} (2): s_2 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \\ (3): s_3 = 85 - 8x_2 \geq 0 & \Rightarrow x_2 \leq 10\frac{5}{8} \end{array}\right\} \Rightarrow \begin{array}{c} s_2 = 0 \text{ when } x_2 = 6 \\ (\text{and } s_3 = 37) \end{array}$$

- s_2 will leave the basis

Change basis through row operations

Eliminate s_2 from the basis let x_2 enter the basis—use row operations:

$$\begin{array}{lllllll|l}
z & -2x_1 & -3x_2 & & & & = 0 & (0) \\
 & x_1 & & +s_1 & & & = 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = 85 & (3) \\
\hline
z & -\frac{9}{5}x_1 & & & +3s_2 & & = 18 & (0)+3\cdot(2) \\
 & x_1 & & +s_1 & & & = 23 & (1)-0\cdot(2) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\
 & \frac{37}{15}x_1 & & & -8s_2 & +s_3 & = 37 & (3)-8\cdot(2)
\end{array}$$

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which one should leave? s_1, x_2, or s_3?

Change basis ... x_1 enters the basis (marginal value > 0)

$$
\begin{array}{llllllcr|l}
z & -\frac{9}{5}x_1 & & & +3s_2 & & = & 18 & (0) \\
 & x_1 & & +s_1 & & & = & 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2) \\
 & \mathbf{\frac{37}{15}x_1} & & & -8s_2 & +s_3 & = & 37 & (3)
\end{array}
$$

The value of x_1 increases until a basic variable reaches the value 0:

$$
\left.\begin{array}{ll}
(1): s_1 = 23 - x_1 \geq 0 & \Rightarrow x_1 \leq 23 \\
(2): x_2 = 6 - \frac{1}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 90 \\
(3): s_3 = 37 - \frac{37}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 15
\end{array}\right\} \Rightarrow \begin{array}{l} s_3 = 0 \text{ when} \\ x_1 = 15 \end{array}
$$

x_1 enters and s_3 leaves the basis: perform row operations:

$$
\begin{array}{llllllcr|l}
z & & & & -2.84s_2 & +0.73s_3 & = & 45 & (0)+(3)\cdot\frac{15}{37}\cdot\frac{9}{5} \\
 & & & s_1 & +3.24s_2 & -0.41s_3 & = & 8 & (1)-(3)\cdot\frac{15}{37} \\
 & & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2)-(3)\cdot\frac{15}{37}\cdot\frac{1}{15} \\
 & x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)\cdot\frac{15}{37}
\end{array}
$$

Change basis ... s_2 enters the basis (marginal value > 0)

$$
\begin{array}{lllllcr|l}
z & & & -2.84s_2 & +0.73s_3 & = & 45 & (0)\\
 & & s_1 & +\mathbf{3.24s_2} & -0.41s_3 & = & 8 & (1)\\
 & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2)\\
x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)
\end{array}
$$

The value of s_2 increases until some basic variable value $= 0$:

$$
\left.\begin{array}{ll}
(1): s_1 = 8 - 3.24s_2 \geq 0 & \Rightarrow s_2 \leq 2.47\\
(2): x_2 = 5 - 1.22s_2 \geq 0 & \Rightarrow s_2 \leq 4.10\\
(3): x_1 = 15 + 3.24s_2 \geq 0 & \Rightarrow s_2 \geq -4.63
\end{array}\right\} \Rightarrow \begin{array}{l} s_1 = 0 \text{ when}\\ s_2 = 2.47\end{array}
$$

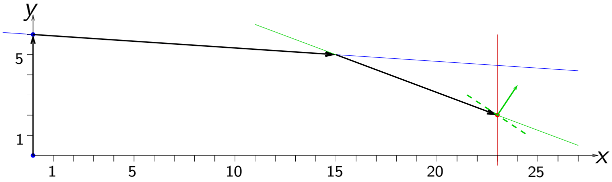
s_2 enters and s_1 leaves the basis: perform row operations

$$
\begin{array}{llllll|cr|l}
z & & & 0.87s_1 & & 0.37s_3 & = & 52 & (0)+(1)\cdot\frac{2.84}{3.24}\\
 & & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 & (1)\cdot\frac{1}{3.24}\\
 & x_2 & & -0.37s_1 & & +0.12s_3 & = & 2 & (2)-(1)\cdot\frac{1.22}{3.24}\\
x_1 & & & +s_1 & & & = & 23 & (3)+(1)
\end{array}
$$

Optimal basic solution

$$
\begin{array}{lllllll}
z & & 0.87 s_1 & & 0.37 s_3 & = & 52 \\
 & & 0.31 s_1 & + s_2 & - 0.12 s_3 & = & 2.47 \\
 & x_2 & - 0.37 s_1 & & + 0.12 s_3 & = & 2 \\
x_1 & & + s_1 & & & = & 23
\end{array}
$$

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- Non-basic variables: $s_1 = s_3 = 0$
- Optimal value: $z = 52$



Summary of the solution course

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-2	-3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	**1**	0	1	0	6
s_3	0	3	8	0	0	1	85
z	1	-1.80	0	0	3	0	18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	**2.47**	0	0	-8	1	37
z	1	0	0	0	-2.84	0.73	45
s_1	0	0	0	1	**3.24**	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
z	1	0	0	0.87	0	0.37	52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Properties of linear minimization (maximization) problems that are utilized for the simplex method

- **Optimality condition**: The *entering* variable in a minimization (maximization) problem should have the largest negative reduced cost (positive marginal value)

  The entering variable *determines a direction* in which the objective value decreases (increases) the fastest

  This direction is *along an edge* of the feasible polyhedron

  If all *reduced costs are positive* (marginal values are negative), then the current basis is *optimal*

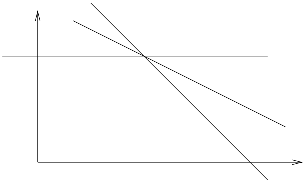
- **Feasibility condition**: The *leaving* variable is the one with smallest nonnegative quotient

  Corresponds to the constraint that would be *violated first*

Degeneracy (Ch. 4.10)

- If the smallest nonnegative quotient (in the feasibility condition) is zero, the value of a basic variable will become zero in the next iteration
- The solution is *degenerate*
- The objective value will *not* improve in this iteration
- Risk: *cycling* around (non-optimal) bases
- Reason: a *redundant* constraint "touches" the feasible set
- Example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 3 \\ x_1 & + & 2x_2 & \leq & 9 \\ & & x_1,\ x_2 & \geq & 0 \end{array}$$



Convergence of the simplex algorithm (Ch. 4.10)

Finite convergence of the simplex algorithm

If all of the basic feasible solutions are non-degenerate, then the simplex algorithm terminates after a finite number of iterations

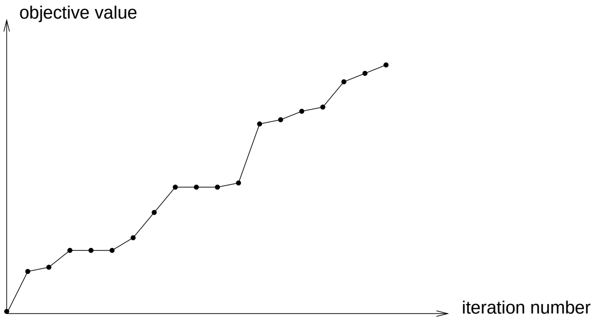
Proof (rough argument)

Non-degeneracy implies that the step length is > 0 in each iteration; hence, we cannot return to an old basic feasible solution once we have left it. Since there are finitely many basic feasible solutions, the algorithm will converge.

- Degeneracy can actually lead to cycling—the same sequence of basic feasible solutions is repeated infinitely
- Remedy: Change the incoming/outgoing criteria!
  E.g., Bland's rule: Sort variables according to some index ordering

Degeneracy

- Typical objective function progress (maximization) of the simplex method



- In modern software: perturb the right hand side $(b_i + \Delta b_i)$ – solve – reduce the perturbation – resolve starting from the current basis – repeat until $\Delta b_i = 0$

Multiple optimal solutions

- If the entering variable has a *zero reduced cost*, then there are (at least) two optimal extreme points
- Also all points on the edge between two optimal extreme points are optimal
  - How does this generalize when there are three or more optimal extreme points?

DRAW GRAPH!!

Unbounded solutions (Ch. 4.4, 4.6)

- If all quotients are *negative*, the value of the variable entering the basis may increase *infinitely*
- Then, the feasible set is *unbounded*
- In a real application this would probably be due to some incorrect assumption (recall "the process of optimization")
- Example:

$$
\begin{aligned}
\text{minimize} \quad z = \quad & -x_1 && -2x_2 && && \text{(3a)} \\
\text{subject to} \quad & -x_1 && +x_2 && \leq 2 && \text{(3b)} \\
& -2x_1 && +x_2 && \leq 1 && \text{(3c)} \\
& && x_1,\ x_2 && \geq 0 && \text{(3d)}
\end{aligned}
$$

DRAW GRAPH!!

Unbounded solutions (Ch. 4.4, 4.6)

- A feasible basis of the optimization problem (3) is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau[1]

basis	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	-5	3	-7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \quad \Longrightarrow \quad s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \quad \Longrightarrow \quad s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely
- The problem has an *unbounded* solution

[1] *Homework: Find this basis using the simplex method*

Find an initial basic feasible solution—phase I

- If an initial basic feasible solution cannot be found easily:
- Assume that $\mathbf{b} \geq \mathbf{0}^m$. Introduce an *artificial variable* a_i in each row that lacks a unit column
- Solve the *phase I-problem*:

$$
\begin{aligned}
\text{minimize} \quad & w = (\mathbf{1}^m)^{\mathrm{T}}\mathbf{a} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}^m\mathbf{a} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}^n, \\
& \mathbf{a} \geq \mathbf{0}^m
\end{aligned}
$$

Find an initial basic feasible solution—phase II

- The case when feasible solutions exist
  - $w^* = 0$, meaning that $\mathbf{a}^* = \mathbf{0}^m$ must hold
  - The resulting basic solution is *optimal in the phase I-problem* and *feasible in the original problem*
  - Start phase-II: solve the original problem, starting from this basic feasible solution
- The case when feasible solutions do not exist
  - $w^* > 0$. The optimal basis then has some $a_i^* > 0$
  - Due to the construction of the objective function, *there exists no feasible solution to the original problem*
  - What to do then? Modelling errors? Can be detected from the optimal solution. In fact, some linear optimization problems are pure feasibility problems