MVE165/MMG631
Linear and integer optimization with applications
Lecture 6
Linear programming: post-optimal and sensitivity analysis

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Derivation of the simplex method (review) (Ch. 4.8)

- B = index set of basic var's, N = index set of non-basic var's

$\Rightarrow$ $|B| = m$ and $|N| = n - m$

- Partition matrix/vectors: $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- The matrix $\mathbf{B}$ ($\mathbf{N}$) contains the columns of $\mathbf{A}$ corresponding to the index set B (N) — Analogously for $\mathbf{x}$ and $\mathbf{c}$

Original linear program

$$\begin{aligned} \text{minimize } z &= \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad \mathbf{A}\mathbf{x} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n \end{aligned}$$

Rewritten linear program

$$\begin{aligned} \text{minimize } z &= \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N \\ \text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N &= \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \; \mathbf{x}_N &\geq \mathbf{0}^{n-m} \end{aligned}$$

Substitute: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \Longrightarrow$

$$\begin{aligned} \text{minimize} \quad z = \quad & [\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N + \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \leq \mathbf{B}^{-1}\mathbf{b}, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Optimality and feasibility (review)

Optimality condition (for minimization)

The basis B is *optimal* if $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
(i.e., reduced costs ≥ 0)

If not, choose as *entering* variable $j^* \in N$ the one with the lowest (negative) value of the reduced cost:

$$j^* = \arg\min_{j \in N} \left\{ c_j - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{A}_j \right\}$$

Feasibility condition

For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_{j^*})_i x_{j^*}$

To stay feasible as x_{j^*} increases from 0, $x_i \geq 0$ must hold $\forall i \in B$

$\Longrightarrow$ Choose the *leaving* variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_{j^*})_i} \;\middle|\; (\mathbf{B}^{-1}\mathbf{A}_{j^*})_i > 0 \right\}$$

The simplex tableau ...

basis	z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
z	1	$\mathbf{0}$	$-(\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})$	$\mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}\mathbf{b}$

... should be interpreted as the system of equations:

$$\begin{aligned} z \quad\quad - \quad (\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N &= \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{x}_B + \quad\quad\quad\quad \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N &= \mathbf{B}^{-1}\mathbf{b} \end{aligned}$$

- We wish to minimize z while also $\mathbf{x}_B \geq \mathbf{0}^m$ and $\mathbf{x}_N \geq \mathbf{0}^{n-m}$ must hold
- For the basis B, it holds that $\mathbf{x}_N = \mathbf{0}^{n-m}$, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, and $z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$

In the simplex tableau, we have

basis	z	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
z	1	$\mathbf{0}$	$-(\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N})$	$\mathbf{c}_B^\top \mathbf{B}^{-1}$	$\mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

- $\mathbf{s}$ denotes possible slack variables [the (blue) columns for $\mathbf{s}$ are *copies* of certain columns for $(\mathbf{x}_B, \mathbf{x}_N)$]
- The computations performed by the simplex algorithm involve matrix inversions (i.e., $\mathbf{B}^{-1}$) and *updates* of these
- A non-basic (basic) variable enters (leaves) the basis $\Rightarrow$ one column, $\mathbf{A}_j$, in $\mathbf{B}$ is replaced by another, $\mathbf{A}_k$, from $\mathbf{N}$
- Row operations $\Leftrightarrow$ Updates of $\mathbf{B}^{-1}$ (and of $\mathbf{B}^{-1}\mathbf{N}$, $\mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c}_B^\top \mathbf{B}^{-1}$)

$\Rightarrow$ Efficient numerical computations are crucial for the performance of the simplex algorithm

Sensitivity analysis—changes in the optimal solution as functions of changes in the problem data (Ch. 5)

- How does the optimum change when the *right-hand-sides* (resources, e.g.) *change*?
- When the *objective coefficients* (prices, e.g.) *change*?

Assume that the basis B is optimal:

$$\begin{array}{ll} \text{minimize} & z = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m}, \end{array}$$

where $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$

Changes in the right-hand-side coefficients

Shadow price = dual price [Def. 5.3]

The *shadow price* of a constraint is defined as the change in the optimal value as a function of the (marginal) change in the RHS. It equals the optimal value of the corresponding dual variable $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}$.

- Suppose $\mathbf{b}$ changes to $\mathbf{b} + \Delta\mathbf{b}$

$\Rightarrow$ New optimal value:

$$\boxed{\mathrm{z}^{\text{new}} = \mathbf{c}_B^\top \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = z + \mathbf{c}_B^\top \mathbf{B}^{-1}\Delta\mathbf{b}}$$

- The current basis is feasible if $\boxed{\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0}$
- If not: negative values will occur in the RHS of the simplex tableau
- The reduced costs are unchanged (positive, at optimum) $\Rightarrow$ resolve using the *dual simplex method* (Ch. 7.3)

Changes in the right-hand-side coefficients

A linear program

$$\begin{array}{lrrrl}
\text{minimize} & z = & -x_1 & -2x_2 & \\
\text{subject to} & & -2x_1 & +x_2 & \leq 2 \\
& & -x_1 & +2x_2 & \leq 7 \\
& & x_1 & & \leq 3 \\
& & & x_1, x_2 & \geq 0
\end{array}$$

Draw graph

The optimal solution is given by

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	-1	-2	-13
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Changes in the right-hand-side coefficients

Change the right-hand-side according to

$$\begin{array}{llrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 + \delta \\ & & x_1 & & \leq 3 \\ & & x_1, x_2 & \geq 0 & \end{array}$$

The change in the RHS is given by $\mathbf{B}^{-1}(0, \delta, 0)^\top = (\frac{1}{2}\delta, 0, -\frac{1}{2}\delta)^\top$ $\Rightarrow$ *new optimal tableau:*

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	-1	-2	$-13 - \delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$5 + \frac{1}{2}\delta$
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$3 - \frac{1}{2}\delta$

- *The current basis is feasible if $-10 \leq \delta \leq 6$ (i.e., if RHS ≥ 0)*

Changes in the right-hand-side coefficients

Suppose $\delta = 8$. The simplex tableau then appears as

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	-1	-2	-21
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	9
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	-1

- Dual simplex iteration: $s_1 = -1$ has to leave the basis
- Find smallest ratio between reduced cost (non-basic column) and (negative) elements in the "s_1-row" (to stay optimal)

s_2 will enter the basis — *new optimal* tableau:

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	-2	0	-5	-19
x_2	0	0	1	1	0	2	8
x_1	0	1	0	0	0	1	3
s_2	0	0	0	-2	1	-3	2

Changes in the objective coefficients

Reduced cost

The *reduced cost* of a non-basic variable defines the change in the objective value when the value of the corresponding variable is (marginally) increased.
The basis B is optimal if $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$ (i.e., reduced costs ≥ 0)

- Suppose $\mathbf{c}$ changes to $\mathbf{c} + \Delta\mathbf{c}$
- The new optimal value:

$$\boxed{\mathrm{z}^{\mathrm{new}} = (\mathbf{c}_B + \Delta\mathbf{c}_B)^\top \mathbf{B}^{-1}\mathbf{b} = z + \Delta\mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}}$$

- The current basis is optimal if

$$\boxed{(\mathbf{c}_N + \Delta\mathbf{c}_N)^\top - (\mathbf{c}_B + \Delta\mathbf{c}_B)^\top \mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}}$$

- If not: more simplex iterations to find the optimal solution

Changes in the objective coefficients

Change the objective according to

$$\begin{array}{rrrrl} \text{minimize} & z = & -x_1 & +(-2+\delta)x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 \\ & & x_1 & & \leq 3 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

The changes in the reduced costs are given by $-(\delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} = (-\frac{1}{2}\delta, -\frac{1}{2}\delta) \Rightarrow$ *new optimal tableau:*

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	$-1+\frac{1}{2}\delta$	$-2+\frac{1}{2}\delta$	$-13+5\delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- *The current basis is optimal if $\delta \leq 2$ (i.e., if reduced costs ≥ 0)*

Changes in the objective coefficients

Suppose $\delta = 4 \Rightarrow$ new tableau:

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	1	0	7
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Let s_2 enter and x_2 leave the basis. New optimal tableau:

basis	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	-2	0	0	-1	-3
s_2	0	0	2	0	1	1	10
x_1	0	1	0	0	0	1	3
s_1	0	0	1	1	0	2	8