MVE165/MMG631
Linear and integer optimization with applications
Lecture 7
Discrete optimization models and applications; complexity

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Recall the diet problem

- Sets
  - $\mathcal{J} = \{1, \ldots, n\}$ — kinds of food
  - $\mathcal{I} = \{1, \ldots, m\}$ — kinds of nutrients
- Variables
  - x_j, $j \in \mathcal{J}$ — purchased amount of food j per day
- Parameters
  - c_j, $j \in \mathcal{J}$ — cost of food j
  - a_j, $j \in \mathcal{J}$ — available amount of food j
  - p_{ij}, $i \in \mathcal{I}$, $j \in \mathcal{J}$ — content of nutrient i in food j
  - q_i — lower limit on the amount of nutrient i per day
  - Q_i — upper limit on the amount of nutrient i per day

The diet problem

The linear optimization model

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & & \displaystyle\sum_{j=1}^{n} c_j x_j, & \\
\text{subject to} & q_i \leq & \displaystyle\sum_{j=1}^{n} p_{ij} x_j \leq Q_i, & i = 1, \dots, m, \\
& 0 \leq & x_j \leq a_j, & j = 1, \dots, n.
\end{array}
$$

- What if we may buy at most $k < n$ different kinds of food?
- Define new variables: $y_j = \begin{cases} 1 & \text{if food } j \text{ is in the diet} \\ 0 & \text{otherwise} \end{cases}$
- Model the following relations:

$$
\begin{array}{lcl}
y_j = 0 & \Longrightarrow & x_j = 0 \\
y_j = 1 & \Longrightarrow & x_j \geq 0
\end{array}
$$

The cardinality constrained diet problem

- Add a *cardinality constraint*: $\displaystyle\sum_{j=1}^{n} y_j \le k$
- Modify the availability constraints: $0 \le x_j \le a_j y_j$

An integer (binary) linear optimization model

$$
\begin{array}{llll}
\underset{x,y}{\text{minimize}} & & \displaystyle\sum_{j=1}^{n} c_j x_j, & \\
\text{subject to} & q_i \le & \displaystyle\sum_{j=1}^{n} p_{ij} x_j \le Q_i, & i = 1, \ldots, m, \\
& & \displaystyle\sum_{j=1}^{n} y_j \le k, & \\
& 0 \le & x_j \le a_j y_j, & j = 1, \ldots, n, \\
& & y_j \in \{0, 1\}, & j = 1, \ldots, n.
\end{array}
$$

The cardinality constrained diet problem—an instance

- Buy at most k types of food
- *Totally n=20 types of food*: SourMilk, Milk, Potato, Carrot, HaricotVerts, GreenBeans, Spinache, Tomato, Cabbage, Banana, Queenberries, OrangeJuice, Chicken, Salmon, Cod, Rice, Pasta, Egg, Apple, Ham
- *Constraints on m=13 nutrients*: Energy, Carbohydrates, Fat, Protein, Fibres, SaturFat, SingleUnsaturFat, MultiUnsaturFat, VitaminD, VitaminC, Folate, Iron, Salt

Optimal solutions for $k \in \{20, 10\}$

k	20	10
Apple	3	3
Banana	2	2
Carrot	2.3	3
Chicken	0.4	– –
Egg	2	2
HaricotVerts	0.1	– –
Milk	3	3
Pasta	2	2
Potato	2.3	2.4
Rice	1	1
Salmon	0.5	0.8
SourMilk	2	2

For $k \leq 9$ no feasible solution exists

Modelling with integer variables (Ch. 13.1)

Variables

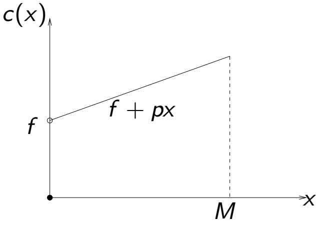
- *Linear programming* (LP) uses continuous variables: $x_{ij} \geq 0$
- *Integer linear programming* (ILP) uses *integer* variables: $x_{ij} \in \mathbb{Z}$
- *Binary linear programming* (BLP) uses *binary* variables: $x_{ij} \in \mathbb{B}$
- If *both* continuous and integer/binary variables are used in a program, it is called a *mixed integer/binary linear program* (MILP)/(MBLP)

Constraints

- An ILP (or MILP) possesses linear constraints and integer requirements on the variables
- Also logical relations, e.g., *if–then* and *either–or*, can be modelled
- This is done by introducing additional (binary) variables and additional constraints

MILP modelling—fixed charges

- Send a truck $\Rightarrow$ Start–up cost: $f > 0$
- Load loafs of bread on the truck $\Rightarrow$ cost per loaf: $p > 0$
- $x = \#$ bread loafs to transport from bakery to store



The cost function $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *nonlinear* and *discontinuos*

$$c(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ f + px & \text{if} \quad 0 < x \leq M \end{cases}$$

MILP modelling—fixed charges

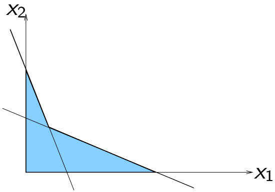
- Let $y = \#$ trucks to send (here, y equals 0 or 1)
- Replace $c(x)$ by $fy + px$
- Constraints: $0 \leq x \leq My$ and $y \in \{0, 1\}$

New model: $\left[\begin{array}{llll} \text{min} & fy + px & & \\ \text{s.t.} & x - My & \leq & 0 \\ & x & \geq & 0 \\ & y & \in & \{0, 1\} \end{array}\right]$

- $y = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad fy + px = 0$
- $y = 1 \quad \Rightarrow \quad x \leq M \quad \Rightarrow \quad fy + px = f + px$
- $x > 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad fy + px = f + px$
- $x = 0 \quad \not\Rightarrow \quad y = 0$ But: Minimization will push y to zero!

Discrete alternatives

- Suppose:
  either $x_1 + 2x_2 \leq 4$ *or* $5x_1 + 3x_2 \leq 10$,
  and $x_1, x_2 \geq 0$ must hold
- *Not* a convex set



Let $M \gg 1$ and define $y \in \{0, 1\}$

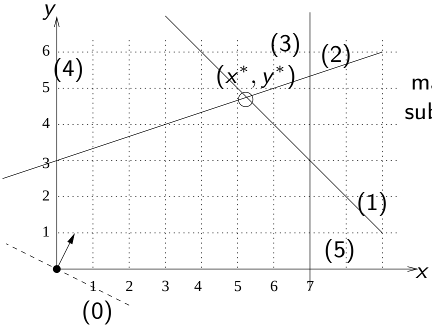
$\Rightarrow$ New set of constraints:
$$\left[\begin{array}{rl} x_1 + 2x_2 \quad - My & \leq 4 \\ 5x_1 + 3x_2 - M(1-y) & \leq 10 \\ y & \in \{0,1\} \\ x_1, x_2 & \geq 0 \end{array}\right]$$

- $y = \begin{cases} 0 & \Rightarrow \quad x_1 + 2x_2 \leq 4 \quad \text{must hold} \\ 1 & \Rightarrow \quad 5x_1 + 3x_2 \leq 10 \quad \text{must hold} \end{cases}$

Exercises: Homework

1. Suppose that you are interested in choosing from a set of investments $\{1, \ldots, 7\}$ using 0/1 variables. Model the following constraints:
    1. You cannot invest in all of them
    2. You must choose at least one of them
    3. Investment 1 cannot be chosen if investment 3 is chosen
    4. Investment 4 can be chosen only if investment 2 is also chosen
    5. You must choose either both investment 1 and 5 or neither
    6. You must choose either at least one of the investments 1, 2 and 3 or at least two investments from 2, 4, 5 and 6
2. Formulate the following as mixed integer progams:
    1. $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$
    2. $v = |x_1 - x_2|$ with $0 \leq x_j \leq C$ for $j = 1, 2$
    3. The set $X \setminus \{x^*\}$ where $X = \{x \in Z^n | Ax \leq b\}$ and $x^* \in X$

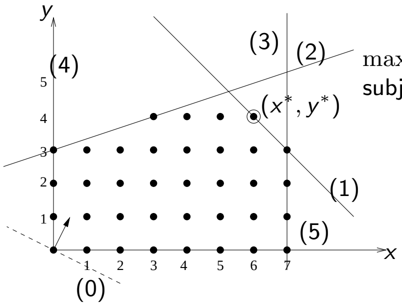
Linear programming: A small example



$$
\begin{array}{lrcrcrl}
\text{maximize} & x & + & 2y & & & (0) \\
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5)
\end{array}
$$

- Optimal solution: $(x^*, y^*) = (5\frac{1}{4}, 4\frac{3}{4})$
- Optimal objective value: $14\frac{3}{4}$

Integer linear programming: A small example

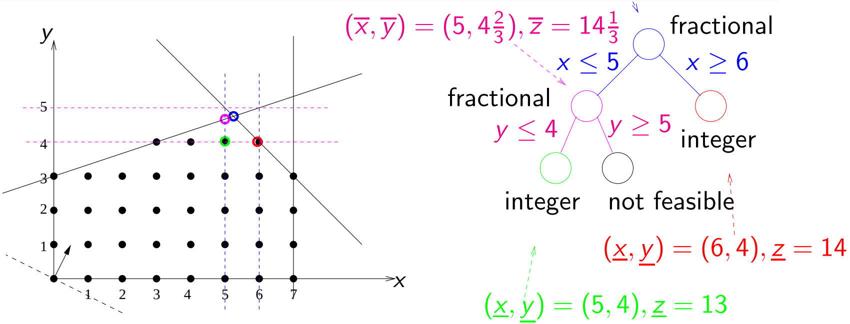


$$
\begin{array}{llcrcll}
\text{maximize} & x & + & 2y & & & (0) \\
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5) \\
 & & & x, y & \text{integer} & &
\end{array}
$$

- What if the variables must take integer values?
- Optimal solution: $(x^*, y^*) = (6, 4)$
- Optimal objective value: $14 < 14\frac{3}{4}$
- The optimal value decreases (possibly constant) when the variables are restricted to possess only integral values

ILP: Solution by the branch–and–bound algorithm (e.g., Gurobi, Cplex, or CLP) (Ch. 15.1–15.2)

- Relax integrality requirements $\Rightarrow$ linear, continuous problem $\Rightarrow$ $(\overline{x}, \overline{y}) = (5\frac{1}{4}, 4\frac{3}{4}), \overline{z} = 14\frac{3}{4}$
- Search tree: branch on fractional variable values



For n binary variables: $\leq 2^n$ branches in the search tree

- Select an optimal collection of objects or investments or projects or ...
  - c_j = benefit of choosing object j, $j = 1, \dots, n$
- Limits on the budget
  - a_j = cost of object j, $j = 1, \dots, n$
  - b = total budget

- Variables: $x_j = \begin{cases} 1, & \text{if object } j \text{ is chosen,} \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, \dots, n$
- Objective function: $\boxed{\max \sum_{j=1}^{n} c_j x_j}$
- Budget constraint: $\boxed{\sum_{j=1}^{n} a_j x_j \leq b}$
- Binary variables: $\boxed{x_j \in \{0, 1\},\ j = 1, \dots, n}$

Computational complexity—the knapsack problem (Ch 2.6)

A small knapsack instance

$$\begin{array}{lrcl} z_1^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & \leq & 89\,643\,482 \\ & x_1, \dots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (0, 1, 2444, 0, 0)$, $z_1^* = 27\,157\,212$
- Cplex finds this solution in 0.015 seconds

The equality version

$$\begin{array}{lrcl} z_2^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & = & 89\,643\,482 \\ & x_1, \dots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (7334, 0, 0, 0, 0)$, $z_2^* = 1\,562\,142$
- Cplex computations interrupted after 1700 sec. ($\approx \frac{1}{2}$ hour)
  - No integer solution found
  - Best upper bound found: 25 821 000
  - 55 863 802 branch–and–bound nodes visited
  - Only *one* feasible solution exists!

The assignment model (Ch. 13.5)

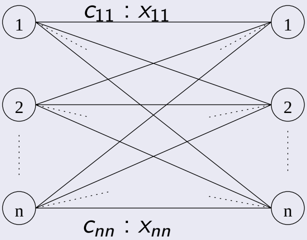
Assign each task to one resource, and each resource to one task

- A cost c_{ij} for assigning task i to resource j, $i, j \in \{1, \ldots, n\}$
- Variables: $x_{ij} = \begin{cases} 1, & \text{if task } i \text{ is assigned to resource } j \\ 0, & \text{otherwise} \end{cases}$

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} & = & 1, & i = 1, \ldots, n \\
 & \displaystyle\sum_{i=1}^{n} x_{ij} & = & 1, & j = 1, \ldots, n \\
 & x_{ij} & \geq & 0, & i, j = 1, \ldots, n
\end{array}
$$

The assignment model

Choose *one* element from each row and each column



c_{11}	c_{12}	c_{13}					c_{1n}
c_{21}	c_{22}	c_{23}					c_{2n}
c_{31}	c_{32}	c_{33}					c_{3n}
c_{n1}	c_{n2}	c_{n3}					c_{nn}

- This integer linear model has *integral extreme points*, since it can be formulated as a network flow problem (Lect. 11–12)
- Therefore, it can be efficiently solved using specialized (network) linear programming techniques
- Even more efficient special purpose (primal–dual–graph-based) algorithms exist

Computational complexity

- Mathematical insight yields efficient algorithms
- E.g., the *assignment problem*
  - $\#$ feasible solutions: $n!$ $\Longrightarrow$ Combinatorial explosion
  - An algorithm $\exists$ that solves this problem in time $\mathcal{O}(n^4) \propto n^4$

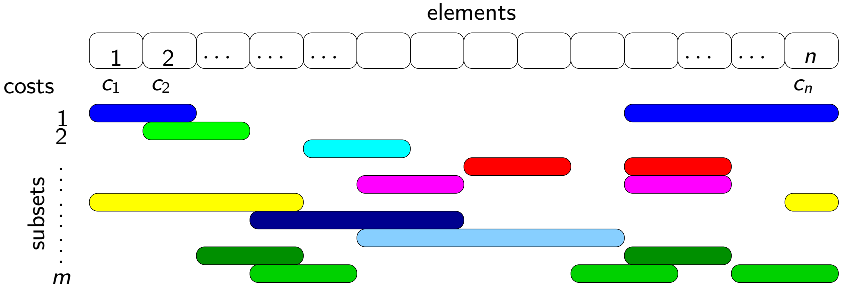
Complete enumeration of all solutions is *not* efficient

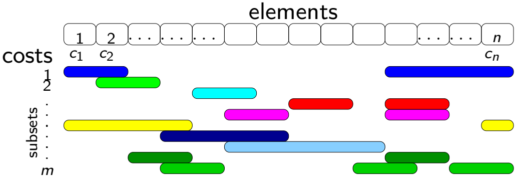
n	2	5	8	10	100	1000
$n!$	2	120	40 000	3 600 000	$9.3 \cdot 10^{157}$	$4.0 \cdot 10^{2567}$
2^n	4	32	256	1 024	$1.3 \cdot 10^{30}$	$1.1 \cdot 10^{301}$
n^4	16	625	4 100	10 000	$1.0 \cdot 10^{8}$	$1.0 \cdot 10^{12}$
$n \log n$	0.6	3.5	7.2	10	200	3 000

- Binary knapsack: $\mathcal{O}(2^n)$
- Continuous knapsack (sorting of $\frac{c_j}{a_j}$): $\mathcal{O}(n \log n)$

Set covering problem—exponential complexity (Ch. 13.8)

- A number (n) of items and a cost for each item
- A number (m) of subsets of the n items
- Find a selection of the items such that each subset contains at least one selected item and such that the total cost for the selected items is minimized





Mathematical formulation

$$\begin{array}{lrcl} \min & \mathbf{c}^\top \mathbf{x} & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \geq & \mathbf{1} \\ & \mathbf{x} & & \text{binary} \end{array}$$

- $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^m$ are constant vectors
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix with entries $a_{ij} \in \{0, 1\}$
- $\mathbf{x} \in \mathbb{R}^n$ is the vector of variables
- Related models: *set partitioning* ($\mathbf{A}\mathbf{x} = \mathbf{1}$), *set packing* ($\mathbf{A}\mathbf{x} \leq \mathbf{1}$)