MVE165/MMG631
Linear and integer optimization with applications
Lecture 9
Discrete optimization: theory and algorithms

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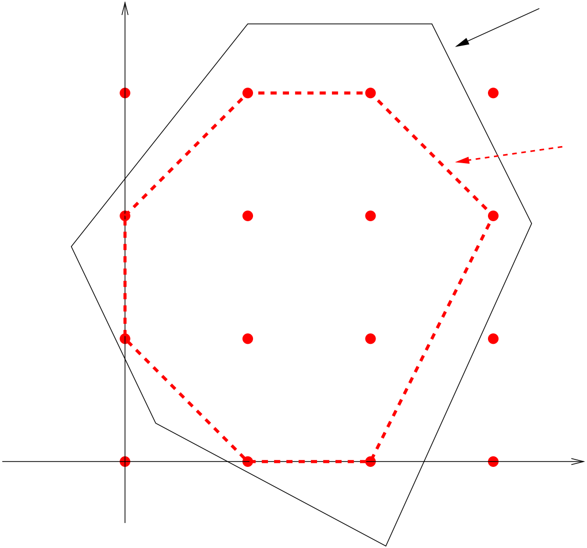
Overview

- Ideal formulation of an integer linear optimization problem
- Relaxations:
  - Cutting planes and algorithms
  - Lagrangean relaxation and Lagrangean duals
- TSP and routing problems
  - Branch–and–bound for structured problems



$\mathbf{A}\mathbf{x} \leq \mathbf{b}$

Ideal since all extreme points are integral

The linear program has integer extreme points

Cutting planes: A very small example

Consider the following ILP:

$$\min\{-x_1 - x_2 : 2x_1 + 4x_2 \leq 7, x_1, x_2 \geq 0 \text{ and integer}\}$$

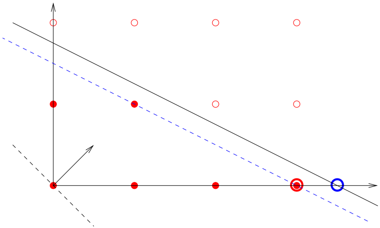
- ILP optimal solution: $z = -3$, $\mathbf{x} = (3, 0)$
- LP (continuous relaxation) optimum: $z = -3.5$, $\mathbf{x} = (3.5, 0)$

Generate a simple cut

"Divide the constraint" by 2 *and round the RHS down*

$x_1 + 2x_2 \leq 3.5 \Rightarrow x_1 + 2x_2 \leq 3$

Adding this cut to the continuous relaxation yields the optimal ILP solution



Cutting planes: valid inequalities (VIs) (Ch. 14.4)

Consider the ILP

$$\begin{array}{llcll} \max & 7x_1 + 10x_2 & & & \\ \text{subject to} & -x_1 + 3x_2 & \leq & 6 & (1) \\ & 7x_1 + x_2 & \leq & 35 & (2) \\ & x_1, x_2 & \geq & 0, \text{ integer} & \end{array}$$

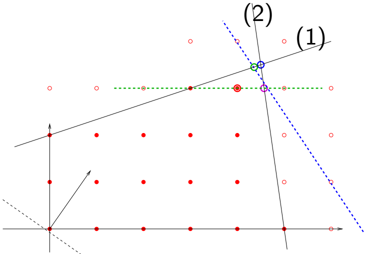
- LP optimum: $z = 66.5$, $\mathbf{x} = (4.5, 3.5)$
- ILP optimum: $z = 58$, $\mathbf{x} = (4, 3)$

Generate a VI:

"Add" the two constraints (1) and (2): $6x_1 + 4x_2 \leq 41 \Rightarrow$ $3x_1 + 2x_2 \leq 20 \Rightarrow \mathbf{x} = (4.36, 3.45)$

Generate another VI:

"$7\cdot(1)+(2)$": $22x_2 \leq 77 \Rightarrow x_2 \leq 3$ $\Rightarrow \mathbf{x} = (4.57, 3)$



Cutting plane algorithms (iterativley better approximations of the convex hull) (Ch. 14.5)

- Choose a suitable mathematical formulation of the problem

A general cutting plane algorithm (cf. p. 378)

1. Solve the linear programming (LP) relaxation
2. If the LP solution is integer: stop; an optimal solution to ILP is found
3. Add one or several *valid inequalities* that *cut off the fractional* solution but *none of the integer solutions*
4. Resolve the new problem and go to step 2

- *Remark:* An inequality in higher dimensions defines a *hyper-plane*; therefore the name cutting *plane*

About cutting plane algorithms

- Problem: It may be necessary to generate VERY MANY cuts
- Each cut should also pass through at least one integer point $\Rightarrow$ faster convergence
- Methods for generating *valid inequalities*
  - Chvátal-Gomory cuts (combine constraints, make beneficial roundings of LHS and RHS)
  - Gomory's method: generate cut from an optimal simplex basis (Ch. 14.5.1)
- Pure cutting plane algorithms are usually less efficient than branch–&–bound
- In commercial solvers (e.g. CPLEX), cuts are used to help (presolve) the branch–&–bound algorithm
- For problems with specific structures (e.g. TSP and set covering) problem specific classes of cuts are used

Step 3 of the cutting plane algorithm – when the linear programming optimal solution is fractional

- Consider the optimal basis B:

$$\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

- For all $i \in B$, defining $\bar{a}_{ij} = (\mathbf{B}^{-1}\mathbf{N})_{ij}$ and $\bar{b}_i = (\mathbf{B}^{-1}\mathbf{b})_i$, then

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i \qquad (1)$$

- Consider an $i \in B$ such that $\bar{b}_i$ is *non-integer* and define the fractions
  - $\tilde{b}_i := \bar{b}_i - \lfloor \bar{b}_i \rfloor \in (0, 1)$
  - $\tilde{a}_{ij} := \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \in [0, 1), \; j \in N$
- From (1) then follows that

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor = \tilde{b}_i - \sum_{j \in N} \tilde{a}_{ij} x_j \qquad (2)$$

Gomory's cutting plane algorithm (Ch. 14.5.1)

- By construction, the LHS of (2), i.e.,

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor \qquad \text{is integer} \tag{3}$$

- Then, also $\tilde{b}_i - \sum_{j \in N} \tilde{a}_{ij} x_j$ must be integer (RHS of (2))
- Since $\tilde{b}_i < 1$, $\tilde{a}_{ij} \geq 0$ and $x_j \geq 0$, $j \in N$, it then follows that

$$\tilde{b}_i - \sum_{j \in N} \tilde{a}_{ij} x_j < 1 \quad \Longrightarrow \quad \tilde{b}_i - \sum_{j \in N} \tilde{a}_{ij} x_j \leq 0$$

- Add the constraint $\sum_{j \in N} \tilde{a}_{ij} x_j - x_{n+1} = \tilde{b}_i$ to the problem
- Since $\tilde{b}_i > 0$ and $x_j = 0$, $j \in N$, it is clear that the *current basic solution becomes infeasible* w.r.t. the new constraint
- But the added constraint does *not cut any integer solutions*

Lagrangian relaxation ($\Rightarrow$ optimistic estimates of z^*) (Ch. 17.1–17.2)

Consider a minimization integer linear program (ILP)

$$
\text{[ILP]} \qquad z^* = \begin{array}{rrcll} \min & \mathbf{c}^\top \mathbf{x} & & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} & (1) \\ & \mathbf{D}\mathbf{x} & \leq & \mathbf{d} & (2) \\ & \mathbf{x} & \geq & \mathbf{0} \text{ and integer} & \end{array}
$$

Assume that the constraints (1) are complicating (subtour eliminating constraints for TSP, e.g.)

- Define the set $X = \{\mathbf{x} \in Z_+^n \,|\, \mathbf{D}\mathbf{x} \leq \mathbf{d}\}$
- Remove the constraints (1) and add them—with penalty parameters $\mathbf{v}$—to the objective function

$$
h(\mathbf{v}) = \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^\top \mathbf{x} + \mathbf{v}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \qquad (3)
$$

Weak duality of Lagrangian relaxations

Theorem

For any $\mathbf{v} \geq \mathbf{0}$ *it holds that* $h(\mathbf{v}) \leq z^*$.

Bevis.

Let $\bar{\mathbf{x}}$ be feasible in [ILP] $\Rightarrow \bar{\mathbf{x}} \in X$ and $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}$. It then holds that

$$h(\mathbf{v}) = \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^\top \mathbf{x} + \mathbf{v}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \underbrace{\leq}_{\bar{\mathbf{x}} \in X} \mathbf{c}^\top \bar{\mathbf{x}} + \mathbf{v}^\top (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) \underbrace{\leq}_{\substack{\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}, \\ \mathbf{v} \geq \mathbf{0}}} \mathbf{c}^\top \bar{\mathbf{x}}.$$

Since an optimal solution $\mathbf{x}^*$ to [ILP] is also feasible, it holds that

$$h(\mathbf{v}) \leq \mathbf{c}^\top \mathbf{x}^* = z^*.$$

□

$\Rightarrow$ $h(\mathbf{v})$ is a *lower bound* on the optimal value z^* for any $\mathbf{v} \geq \mathbf{0}$

The best lower bound is given by

$$h^* = \max_{\mathbf{v} \geq \mathbf{0}} h(\mathbf{v}) = \max_{\mathbf{v} \geq \mathbf{0}} \left\{ \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^\top \mathbf{x} + \mathbf{v}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \right\} \leq z^*$$

Tractable Lagrangian relaxations

- Special algorithms for maximizing the Lagrangian dual function h exist (e.g., subgradient optimization, Ch. 17.3)
- h is always concave but typically nondifferentiable
- For each value of $\mathbf{v}$ chosen, a subproblem (3) must be solved
- For general ILP's: typically a non-zero duality gap: $z^* - h^* > 0$
- The Lagrangian relaxation bound is never worse that the linear programming relaxation bound: $z^{\mathrm{LP}} \leq h^* \leq z^*$
- If the set X has the integrality property (i.e., X^{LP} has integral extreme points), then $h^* = z^{\mathrm{LP}}$
- Choose the constraints ($\mathbf{Ax} \leq \mathbf{b}$) to dualize such that the relaxed problem (3) is computationally tractable but still does not possess the integrality property

An ILP Example

[HOMEWORK]

Find optimistic and pessimistic bounds for the following ILP example using the branch–&–bound algorithm, a cutting plane algorithm, and Lagrangean relaxation.

$$
\begin{array}{lrcl}
\max & 5x_1 + 4x_2 & & \\
\text{s.t.} & x_1 + x_2 & \leq & 5 \\
& 10x_1 + 6x_2 & \leq & 45 \\
& x_1, x_2 & \geq & 0 \text{ and integer}
\end{array}
$$

The linear programming optimal solution is given by the basis $\mathbf{x}_B = \{x_1, x_2\}$ with optimal values $z = 23.75$, $x_1 = 3.75$ and $x_2 = 1.25$

ILP formulation of the TSP problem

- d_{ij}: distance from city i to city j
- Binary variables x_{ij} for each connection
- $V = \{1, \ldots, n\}$: set of nodes (cities)

$$
\begin{array}{llcll}
\min & \sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij}, & & & (0) \\
\text{s.t.} & \sum_{j \in V} x_{ij} = & 1, & i \in V, & (1) \\
& \sum_{i \in V} x_{ij} = & 1, & j \in V, & (2) \\
& \sum_{i \in U, j \in V \setminus U} x_{ij} \geq & 1, & \forall U \subset V : 2 \leq |U| \leq |V| - 2, & (3) \\
& x_{ij} \in & \{0, 1\}, & i, j \in V & (4)
\end{array}
$$

- (0)–(2), (4): assignment problem
- Enter and leave each city exactly once $\Leftrightarrow$ (1) and (2)
- Constraints (3): subtour elimination

Solution methods for the TSP Problem

- Tailored branch–&–bound (Ch. 15)
- Heuristics
  - Constructive heuristics (Ch. 16.3)
  - Local search heuristics (Ch. 16.4)
  - Approximation algorithms (Ch. 16.6)
  - Metaheuristics (Ch. 16.5)
- Lagrangean relaxation and subgradient optimization (Ch. 17).
- Common difficulty for all solution methods for the TSP: Combinatorial explosion: # possible tours $\approx n!$

$\Leftrightarrow$ Very many subtour elimination constraints (3)

Branch–and–bound algorithm for TSP (Ch. 15.4.2)

- Relaxing just the binary constraints (4) in TSP does not yield a tractable problem, since the number of subtour elimination constraints (3) is very large

$\Rightarrow$ An LP with very many constraints

- Relaxing the subtour eliminating constraints (3) yields an assignment problem, having the integrality property, which can be solved in polynomial time
- Solutions to a relaxed problem typically contains a number of sub-tours
- Branch on these sub-tours (rather than on fractional variables)
- Branching $\Leftrightarrow$ partitioning of the solution space

DRAW AN EXAMPLE