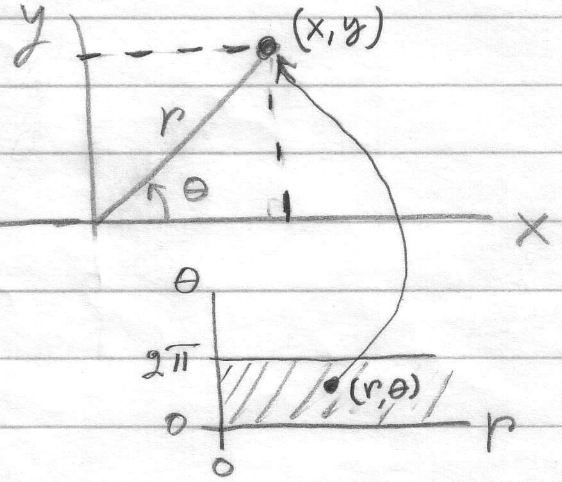
14.4 Polära koordinater

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \operatorname{atan}(y/x) \end{cases}$$

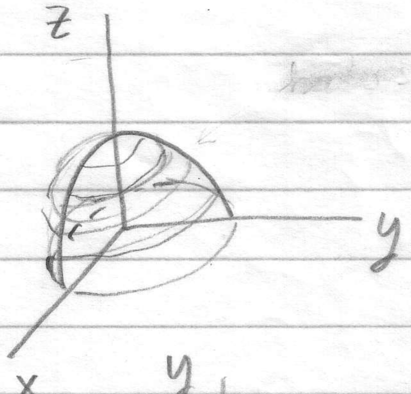


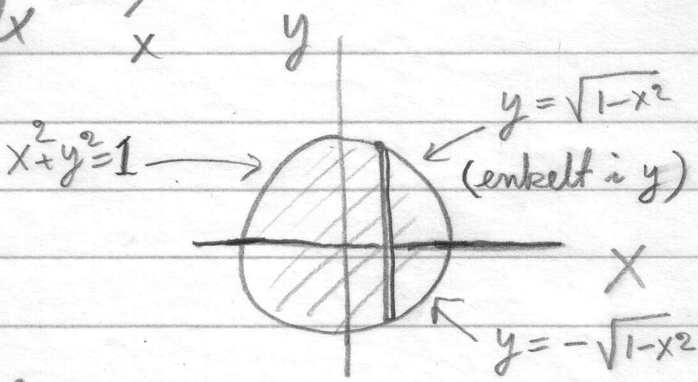
Exempel. Volymen mellan xy-planet och $z = 1 - x^2 - y^2$.

Volymen under grafen:

$$V = \iint_{x^2+y^2\le 1} (1 - x^2 - y^2)\,dA =$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx$$

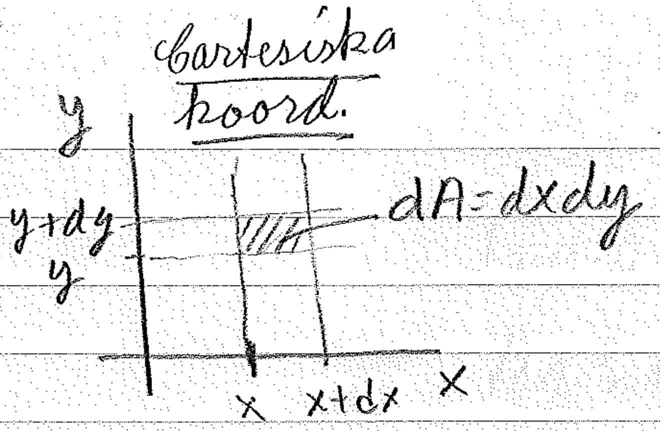


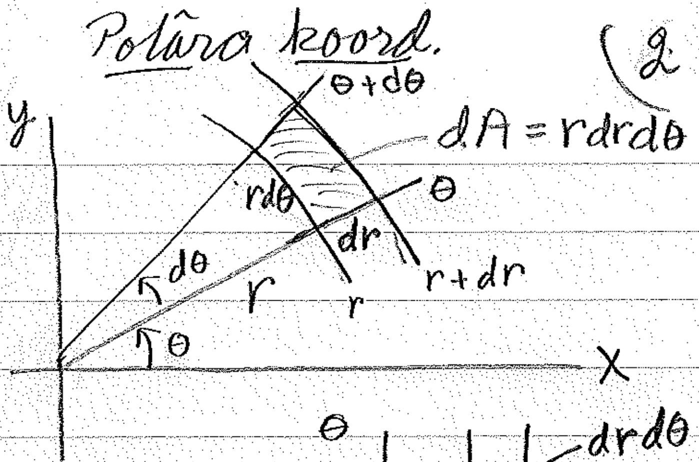


Mycket svår.

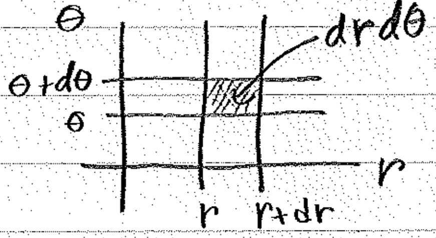
Polära koordinater:

$$V = \iint_{r\le 1} (1 - r^2)\,dA$$

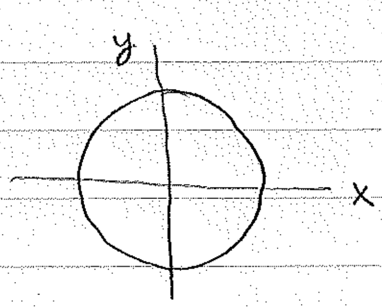




$$dx\,dy = dA = r\,dr\,d\theta$$



$$V = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta =$$



$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

Exempel 4 Visa att $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

Bevis: Låt $I = \int_{-\infty}^{\infty} e^{-x^2}dx$. Då blir

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2}dA$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\,r\,dr$$

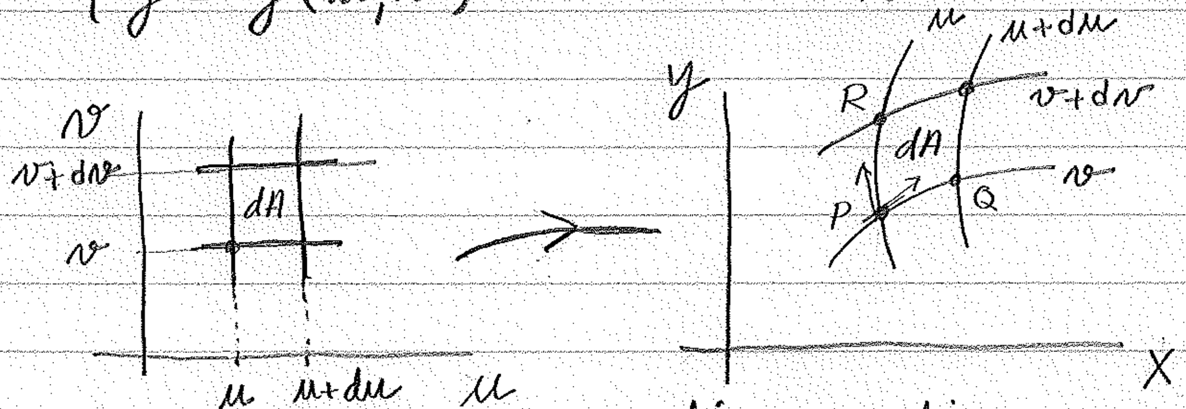
$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty = 2\pi \cdot \frac{1}{2} = \pi$$

$$\Rightarrow \quad I = \sqrt{\pi}.$$

Variabelbyte. Nya koordinater (u, v).

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

(Gamla koord. x, y som funktion av de nya u, v.)



koordinat-kurvor

Den inversa transformationen finns om jacobi-determinanten $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$

$dA = |\overrightarrow{PQ} \times \overrightarrow{PR}|$ (sid 582)

$$\overrightarrow{PQ} = dx\,\bar{i} + dy\,\bar{j} = \{\text{kedjeregeln}\}$$

$$= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} \underbrace{dv}_{=0}\right)\bar{i} + \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} \underbrace{dv}_{=0}\right)\bar{j}$$

$$= \frac{\partial x}{\partial u} du\,\bar{i} + \frac{\partial y}{\partial u} du\,\bar{j} = \underbrace{\left(\frac{\partial x}{\partial u}\bar{i} + \frac{\partial y}{\partial u}\bar{j}\right)}_{\text{tangent-vektor i } P} du$$

$$\overrightarrow{PR} = \underbrace{\left(\frac{\partial x}{\partial v}\bar{i} + \frac{\partial y}{\partial v}\bar{j}\right)}_{\text{tangentvektor i } P} dv$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{vmatrix} =$$

$$= \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{transponatet}} du\,dv\,\bar{k} = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,\bar{k}$$

av Jacobi-matrisen $\quad \det(A^T) = \det(A)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

← determinant

längd $\qquad$ belopp

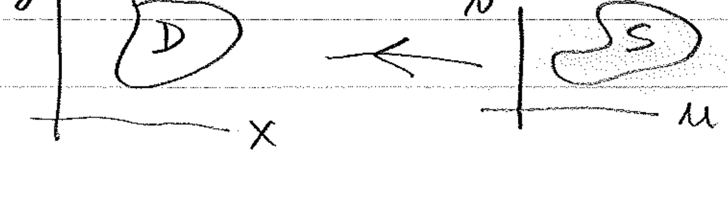
$$dA = |\overrightarrow{PQ}\times\overrightarrow{PR}| = \left|\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,\bar{k}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

dvs $\boxed{dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$

dvs arean skalas med absolut-beloppet av Jacobi-determinanten.

<u>Sats 4</u> (Variabelbyte i dubbelintegral)

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



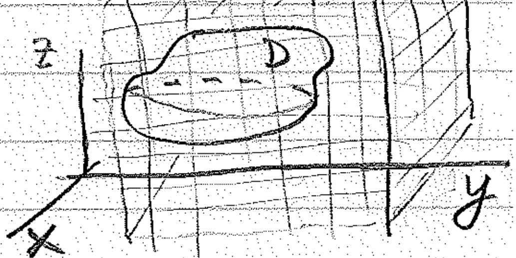
Exempel $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$dA = |r|\, dr\, d\theta = r\, dr\, d\theta$$

14.5 Trippelintegralen



$$\iiint_D f(x,y,z)\, dV = \iiint_D f(x,y,z)\, dx\, dy\, dz$$

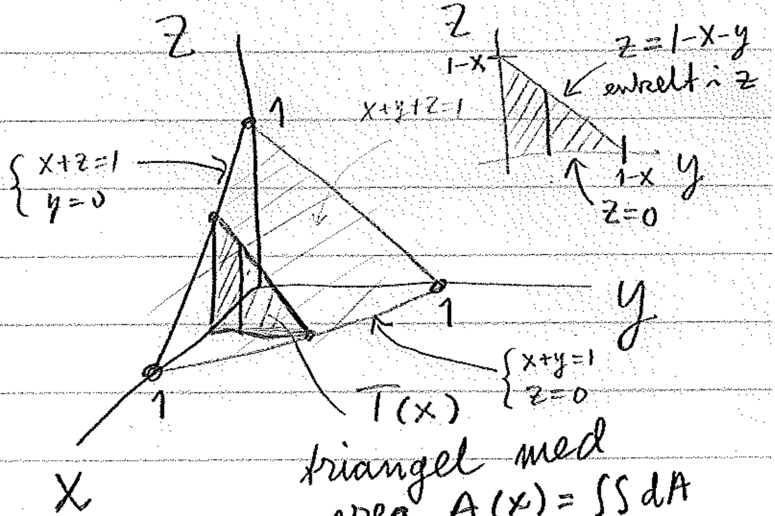
Definieras med partition och Riemann-summa. Kan ibland beräknas med upprepad integration.

Exempel Volymen av tetraeder.

$$V = \iiint_T dV = \int_0^1 A(x)\, dx$$

$$= \int_0^1 \iint_{T(x)} dA\, dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx =$$



triangel med area $A(x) = \iint_{T(x)} dA$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\,dy\,dx = \int_{x=0}^{1} \left[(1-x)y - \tfrac{1}{2}y^2\right]_{y=0}^{1-x} dx$$

$$= \int_0^1 \left((1-x)^2 - \tfrac{1}{2}(1-x)^2\right) dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{2}\left[\frac{-(1-x)^3}{3}\right]_0^1$$

$$= \frac{1}{6}.$$