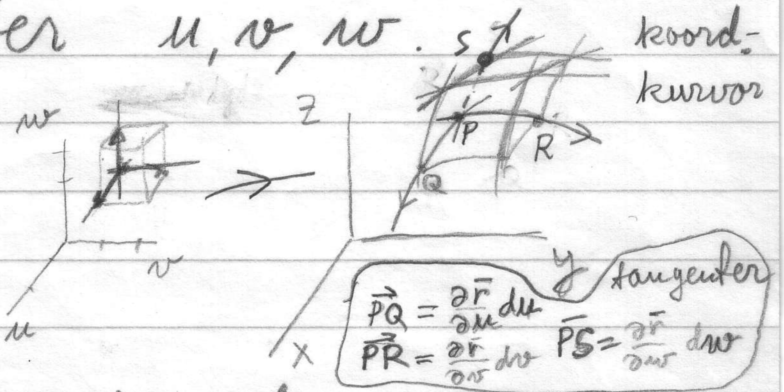
FÖ 5.1 (2007-08)

14.6 Variabelbyte i trippelintegralen.

Nya koordinater u, v, w.

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$



Om Jacobi-determinanten är $\neq 0$ så blir volymselementet

$dV = |\vec{PQ} \bullet (\vec{PR} \times \vec{PS})|$ sid 582

$$dV = dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

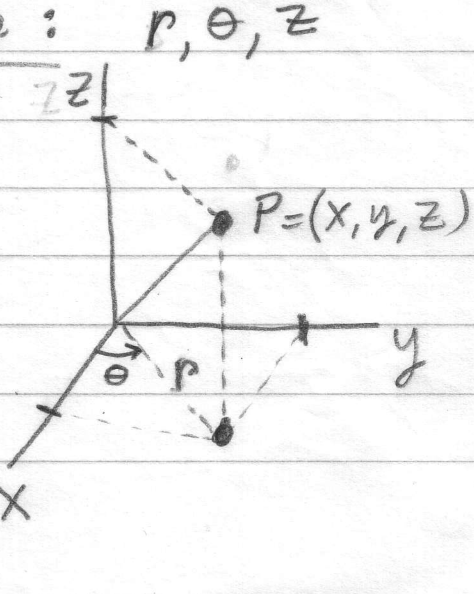
(absolutbeloppet av Jacobi-determinanten)

Två speciellt viktiga koordinatsystem:

cylindriska och sfäriska.

Cylindriska koordinater: r, θ, z

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$0 \le r < \infty$

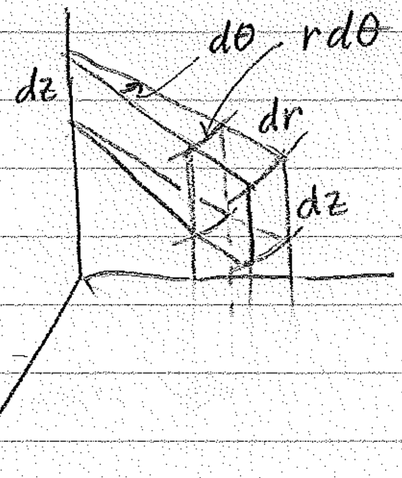
$0 \le \theta < 2\pi$

$-\infty < z < \infty$

Jacobi-determinanten:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

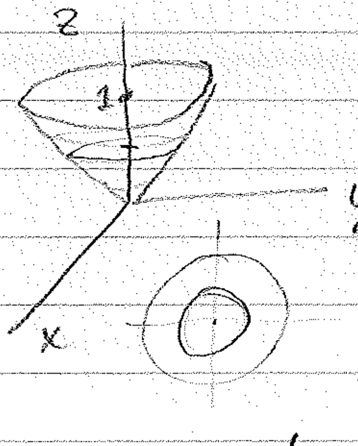


Volymselementet:

$$dV = r\,dr\,d\theta\,dz$$

Exempel Beräkna $I = \iiint_K (x^2+y^2)\,dx\,dy\,dz$

över konen $K: \begin{cases} 0 \le z \le 1 \\ \sqrt{x^2+y^2} \le z \end{cases}$

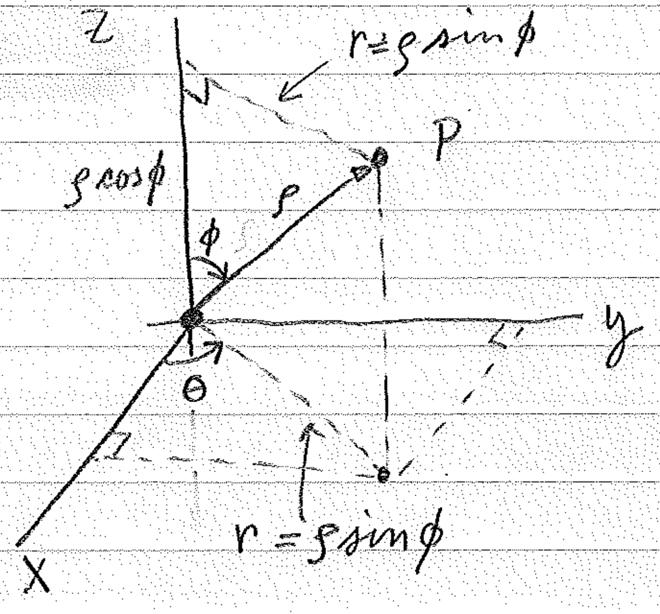


$$I = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{z} r^2\, r\,dr\,d\theta\,dz$$

$$= \int_0^1 \int_0^z r^3\,dr\,dz \cdot \int_0^{2\pi} d\theta = \int_0^1 \frac{z^4}{4}\,dz \cdot 2\pi = 2\pi \cdot \frac{1}{20}$$

Sfäriska koordinater (ρ, ϕ, θ)

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$



$\rho = \sqrt{x^2 + y^2 + z^2}$

$r = \rho \sin\phi$

$\tan\phi = \dfrac{r}{z} = \dfrac{\sqrt{x^2+y^2}}{z}$

$0 \le r < \infty$

$0 \le \phi \le \pi$

$0 \le \theta < 2\pi$

$\tan\theta = \dfrac{y}{x}$

$\phi = 0$ är en radie mot nordpolen, $\phi = \pi$ är en radie mot sydpolen

$\phi = \pi/2$ är ekvatorn

Jacobi-determinanten:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} =$$

$$= \sin\phi\cos\theta \cdot \rho^2\sin^2\phi\cos\theta + \rho\cos\phi\cos\theta\,\rho\sin\phi\cos\phi\cos\theta$$
$$-\rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)$$
$$= \{\text{trig. ettan}\} = \rho^2\sin\phi$$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \quad (\text{obs: } \sin\phi \geq 0)$

Exempel. Tröghetsmomentet för enhetsklotet m.o.p. z-axeln: $\left(\delta = \text{masstäthet } \left[\frac{kg}{m^3}\right]\right)$

$$I = \iiint_B (x^2+y^2)\delta \, dV = \delta \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (\rho \sin\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \delta \int_0^1 \rho^4 d\rho \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos\phi \\ du = +\sin\phi \, d\phi \\ \sin^2\phi = 1 - u^2 \\ \phi = 0 \Rightarrow u = -1, \phi = \pi \Rightarrow u = 1 \end{array} \right\}$$

$$= \delta \frac{1}{5} \cdot \int_{-1}^{1} (1-u^2)\, du \cdot 2\pi = \delta \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15}\delta$$

δ = masstäthet = konstant $\left[\frac{kg}{m^3}\right]$

14.7 Endast "Moments and Centres of Mass".
sid 798–801

Masscentrum: $(\bar{x}, \bar{y}, \bar{z})$ där

$$\bar{x} = \frac{\iiint_R x\,\delta(x,y,z)\,dV}{\iiint_R \delta(x,y,z)\,dV}$$

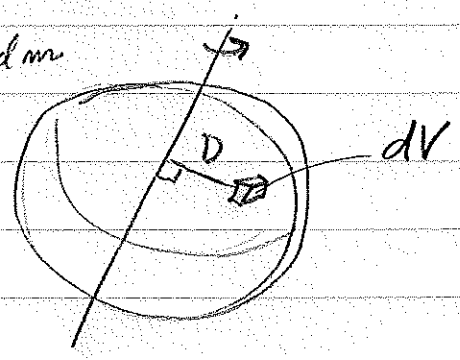
δ = masstäthet = = densitet $[kg/m^3]$

osv.

På vektorform: $$\bar{\mathbb{r}} = \frac{\iiint_R \mathbb{r}\,\delta\,dV}{\iiint_R \delta\,dV} = \frac{\iiint_R \mathbb{r}\,dm}{\iiint_R dm}$$

Tröghetsmoment m.a.p. axel:

$$I = \iiint_R D^2(x,y,z)\,\delta(x,y,z)\,dV = \iiint_R D^2\,dm$$



D = avståndet till axeln