FÖ 6.2

Ytintegral och flödesintegral

S: $\bar{r} = \bar{r}(u,v)$, $(u,v) \in D$

- tangenter: $\frac{\partial \bar{r}}{\partial u}$, $\frac{\partial \bar{r}}{\partial v}$
- normalvektor: $\bar{N} = \pm \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ (in/ut, upp/ner)
- enhetsnormal: $\hat{N} = \frac{\bar{N}}{|\bar{N}|}$
- ytelement: $dS = \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\,dv = |\bar{N}|\, du\,dv$
- normalytelement:

$$d\bar{S} = \pm \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\, du\,dv = \bar{N}\, du\,dv = \hat{N}\, dS$$

Exempel Sfär: $x^2 + y^2 + z^2 = R^2$

Sfäriska koordinater: $\rho = R$

$$\begin{cases} x = R \sin\phi \cos\theta \\ y = R \sin\phi \sin\theta \\ z = R \cos\phi \end{cases} \qquad \begin{matrix} \theta \in [0, 2\pi] \\ \phi \in [0, \pi] \end{matrix}$$

$$\frac{\partial \bar{r}}{\partial \phi} = R\cos\phi\cos\theta\, \bar{i} + R\cos\phi \sin\theta\, \bar{j} - R\sin\phi\, \bar{k}$$

$$\frac{\partial \bar{r}}{\partial \theta} = -R\sin\phi\sin\theta\, \bar{i} + R\sin\phi\cos\theta\, \bar{j}$$

$$\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= R^2\sin\phi \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

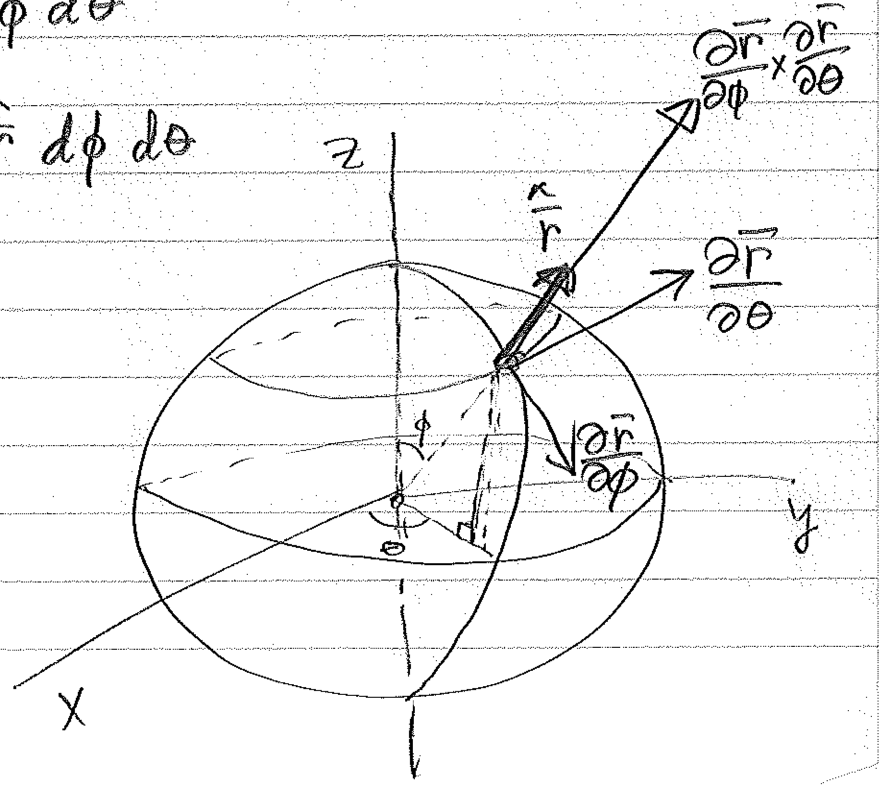
$$= R^2\sin\phi \underbrace{\left(\sin\phi\cos\theta\,\bar{i} + \sin\phi\sin\theta\,\bar{j} + \cos\phi\,\bar{k} \right)}_{= \frac{\bar{r}}{R} = \frac{\bar{r}}{|\bar{r}|} = \hat{r}}$$

$$= R^2 \sin\phi\ \hat{r}$$

$$\left|\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta}\right| = R^2 \sin\phi \qquad \left(\text{obs: } \sin\phi \geq 0 \right)$$

$$dS = R^2 \sin\phi \, d\phi \, d\theta$$

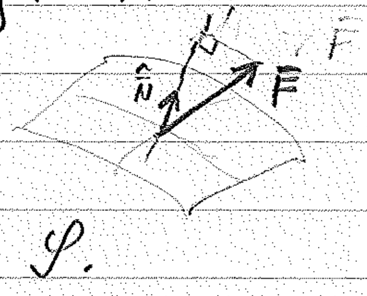
$$d\bar{S} = \pm R^2 \sin\phi \ \hat{r} \, d\phi \, d\theta$$



Ytintegral: $\iint_{\mathcal{S}} f\, dS = \iint_{D} f(\bar{r}(u,v)) \left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right| du\,dv$

Flödesintegral: $\iint_{\mathcal{S}} \bar{F} \bullet d\bar{S} = \iint_{\mathcal{S}} \bar{F} \bullet \hat{N}\, dS$

Normalkomponent: $\bar{F} \bullet \hat{N}$



Exempel Massflöde genom $\mathcal{S}$.

$\bar{v}$ hastighetsfält $\left[\frac{m}{s}\right]$

δ densitet $\left[\frac{kg}{m^3}\right]$

$\bar{F} = \delta \bar{v}$ massflödestäthet $\left[\frac{kg}{m^2 s}\right]$

Flödet: $\iint_{\mathcal{S}} \bar{F} \bullet d\bar{S} = \iint_{\mathcal{S}} \bar{F} \bullet \hat{N}\, dS$

$$= \iint_{\mathcal{S}} \underbrace{\delta \bar{v}}_{\left[\frac{kg}{m^2 s}\right]} \bullet \hat{N}\, \underbrace{dS}_{[m^2]} \quad \left[\frac{kg}{s}\right]$$

Flödesintegralen kräver att det finns ett entydigt normalvektorfält $\bar{N}$ på $\mathcal{S}$, dvs en orienterbar yta. Möbius band går inte!

FÖ 6.1

Grad, div, rot.

16.1 Grad, div, rot.

endast sid 888 - 892 + Exempel 3, 5

Med nabla-operatorn

$$\bar{\nabla} = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

kan vi bilda

gradienten av skalärt fält:

$$\bar{\nabla}\phi = \overline{\mathrm{grad}}\,\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

och divergensen och rotationen ("curl") av vektorfält:

$$\bar{\nabla}\cdot\bar{F} = \overline{div}\,\bar{F} =$$

$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot\left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\bar{\nabla}\times\bar{F} = \overline{rot}\,\bar{F} = \overline{curl}\,\bar{F}$$

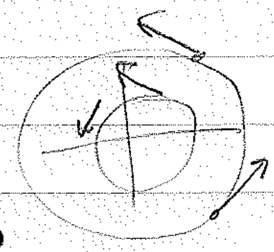
$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

Jag skriver $\bar{\nabla}\phi$, $\bar{\nabla}\cdot\bar{F}$, $\bar{\nabla}\times\bar{F}$, aldrig grad div och rot.

Exempel 5 Stelkroppsrotation



$$\bar{v} = \Omega\left(-y\,\bar{i} + x\,\bar{j}\right)$$

$$\bar{\nabla}\cdot\bar{v} = \Omega\left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x\right) = 0$$

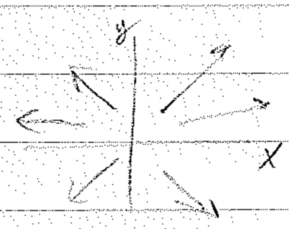
$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 0\bar{i} + 0\bar{j} + 2\Omega\bar{k}$$

$$= 2\Omega\bar{k}$$

Divergerar inte, men roterar.

Exempel

$$\bar{F} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$$



$$\bar{\nabla}\cdot\bar{F} = 1 + 1 + 1 = 3$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Divergerar, Roterar ej.

exempel 3

$$\bar{F} = m\frac{\bar{r}}{\rho^3} = \frac{m}{\rho^3}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\frac{\partial F_1}{\partial x} = m\frac{\frac{\partial x}{\partial x}\rho^3 - x\frac{\partial \rho^3}{\partial x}}{\rho^6} = m\frac{\rho^3 - x\cdot 3\rho^2\frac{\partial\rho}{\partial x}}{\rho^6}$$

$$= m\frac{\rho^3 - x\cdot 3\rho^2\cdot\frac{x}{\rho}}{\rho^6}$$

$$= m\frac{\rho^2 - 3x^2}{\rho^5}$$

$$\boxed{\begin{aligned}\rho &= \sqrt{x^2+y^2+z^2} \\ \frac{\partial\rho}{\partial x} &= \frac{2x}{2\sqrt{x^2+y^2+z^2}} \\ &= \frac{x}{\rho}\end{aligned}}$$

På samma vis

$$\frac{\partial F_2}{\partial y} = m\frac{\rho^2 - 3y^2}{\rho^5}$$

$$\frac{\partial F_3}{\partial z} = m\frac{\rho^2 - 3z^2}{\rho^5}$$

så att (för $\rho \neq 0$)

$$\bar{\nabla}\cdot\bar{F} = m\frac{3\rho^2 - 3(x^2+y^2+z^2)}{\rho^5} = 0$$

Utflödet genom sfär med radie R, $S = \{(x,y,z) : \rho = R\}$

$$\iint_S \bar{F}\cdot\hat{N}\,dS = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} m\underbrace{\frac{\bar{r}}{\rho^3}\cdot\frac{\bar{r}}{\rho}}_{=\frac{\rho^2}{\rho^4}=\frac{1}{\rho^2}=\frac{1}{R^2}} R^2\sin\phi\,d\phi\,d\theta = m\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta$$

$$= 4\pi m$$

$$\begin{cases} dS = R^2\sin\phi\,d\phi\,d\theta \\ \hat{N} = \frac{\bar{r}}{|\bar{r}|} = \hat{r}\end{cases}$$

16.2 Räkneregler

Sats 3

a) $\bar{\nabla}(\phi\psi) = \psi\,\bar{\nabla}\phi + \phi\,\bar{\nabla}\psi$

b) $\bar{\nabla}\cdot(\phi\bar{F}) = \bar{\nabla}\phi\cdot\bar{F} + \phi\,\bar{\nabla}\cdot\bar{F}$

c) $\bar{\nabla}\times(\phi\bar{F}) = \bar{\nabla}\phi\times\bar{F} + \phi\,\bar{\nabla}\times\bar{F}$

d) $\bar{\nabla}\cdot(\bar{F}\times\bar{G}) = (\bar{\nabla}\times\bar{F})\cdot\bar{G} - \bar{F}\cdot(\bar{\nabla}\times\bar{G})$

e) $\bar{\nabla}\times(\bar{F}\times\bar{G}) =$ se boken

f) $\bar{\nabla}(\bar{F}\cdot\bar{G}) =$ se boken

g) $\nabla\cdot(\bar{\nabla}\times\bar{F}) = 0$

h) $\bar{\nabla}\times(\bar{\nabla}\phi) = \bar{0}$

i) $\bar{\nabla}\times(\bar{\nabla}\times\bar{F}) = \nabla(\bar{\nabla}\cdot\bar{F}) - \Delta\bar{F}$

Bevis av a, b, c, d, g, h.
Resten utan bevis.

(b) $\bar{\nabla}\cdot(\phi F_1\bar{i} + \phi F_2\bar{j} + \phi F_3\bar{k}) = \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3)$

$$= \frac{\partial\phi}{\partial x}F_1 + \phi\frac{\partial F_1}{\partial x} + \frac{\partial\phi}{\partial y}F_2 + \phi\frac{\partial F_2}{\partial y} + \frac{\partial\phi}{\partial z}F_3 + \phi\frac{\partial F_3}{\partial z}$$

$$= \bar{\nabla}\phi\cdot\bar{F} + \phi\,\bar{\nabla}\cdot\bar{F}$$

Obs: $\bar{F} = \bar{\nabla}\phi \overset{(h)}{\Longrightarrow} \bar{\nabla} \times \bar{F} = \bar{\nabla} \times \bar{\nabla}\phi = \bar{0}$.

Alltså: ett konservativt fält är rotationsfritt. Detta är det nödvändiga villkoret från 15.2.

Villkoret $\bar{\nabla} \times \bar{F} = \bar{0}$ är tillräckligt villkor enligt Sats 4.

Sats 4 Antag 1) $\bar{\nabla} \times \bar{F} = \bar{0}$ i D
2) D enkelt sammanhängande

Då är $\bar{F}$ konservativt i D, dvs det finns en potential ϕ så att $\bar{\nabla}\phi = \bar{F}$ i D.

Detta betyder att om $\bar{\nabla} \times \bar{F} = \bar{0}$ så är följande system av PDE lösbart:

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = F_1 \\ \dfrac{\partial \phi}{\partial y} = F_2 \\ \dfrac{\partial \phi}{\partial z} = F_3 \end{cases}$$

Exempel. $\bar{F} = \bar{r}$. Då är $\bar{\nabla} \times \bar{F} = \bar{\nabla} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{0}$.

Vi löser $\begin{cases} \dfrac{\partial \phi}{\partial x} = x \\ \dfrac{\partial \phi}{\partial y} = y \\ \dfrac{\partial \phi}{\partial z} = z \end{cases} \Rightarrow \begin{cases} \phi = \frac{1}{2}x^2 + f(y,z) \\ \phi = \frac{1}{2}y^2 + g(x,z) \\ \phi = \frac{1}{2}z^2 + h(x,y) \end{cases}$ Tag $\begin{cases} f(y,z) = \frac{1}{2}(y^2+z^2)+C \\ g(x,z) = \frac{1}{2}(x^2+z^2)+C \\ h(x,y) = \frac{1}{2}(x^2+y^2)+C \end{cases}$

$$\phi(x,y,z) = \frac{1}{2}(x^2+y^2+z^2) + C = \frac{1}{2}|\bar{r}|^2 + C = \frac{1}{2}r^2 + C$$