FÖ 7.1

Kurvintegraler

$C: \quad \mathbb{r} = \mathbb{r}(t), \quad t \in [a, b]$

Tangent: $\mathbb{T} = \mathbb{r}'(t)$, enhetstangent: $\hat{\mathbb{T}} = \dfrac{\mathbb{T}}{|\mathbb{T}|}$

$$ds = |\mathbb{r}'(t)|\, dt$$

$$d\mathbb{r} = \hat{\mathbb{T}}\, ds = \mathbb{r}'(t)\, dt$$

$$\int_C f\, ds = \int_a^b f(\mathbb{r}(t))\, |\mathbb{r}'(t)|\, dt; \quad \int_C \mathbb{F} \cdot d\mathbb{r} = \int_C \mathbb{F} \cdot \hat{\mathbb{T}}\, ds =$$
$$= \int_a^b \mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{r}'(t)\, dt$$

Ytintegraler

$\mathcal{S}: \quad \mathbb{r} = \mathbb{r}(u, v), \quad (u, v) \in D$

Tangenter: $\dfrac{\partial \mathbb{r}}{\partial u}, \dfrac{\partial \mathbb{r}}{\partial v}$

Normalvektor: $\mathbb{N} = \pm \dfrac{\partial \mathbb{r}}{\partial u} \times \dfrac{\partial \mathbb{r}}{\partial v}$ ($\pm$, upp/ner)

Enhetsnormal: $\hat{\mathbb{N}} = \mathbb{N}/|\mathbb{N}|$

$$dS = \left| \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} \right| du\, dv$$

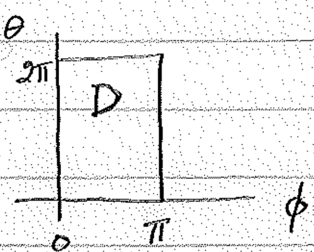
$$d\mathbb{S} = \hat{\mathbb{N}}\, dS = \pm \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}\, du\, dv$$

$$\iint_{\mathcal{S}} f\, dS; \quad \iint_{\mathcal{S}} \mathbb{F} \cdot d\mathbb{S} = \iint_{\mathcal{S}} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS$$

<u>Exempel</u> Sfär $x^2+y^2+z^2=R^2$

Sfäriska koordinater ($\rho=R$):

$$\begin{cases} x = R\sin\phi\cos\theta, & \phi\in[0,\pi] \\ y = R\sin\phi\sin\theta, & \theta\in[0,2\pi] \\ z = R\cos\phi, & \end{cases}$$



$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial\phi} = R\cos\phi\cos\theta\,\mathbf{i} + R\cos\phi\sin\theta\,\mathbf{j} - R\sin\phi\,\mathbf{k} \\ \dfrac{\partial \mathbf{r}}{\partial\theta} = -R\sin\phi\sin\theta\,\mathbf{i} + R\sin\phi\cos\theta\,\mathbf{j} \end{cases}$$

$$\frac{\partial \mathbf{r}}{\partial\phi}\times\frac{\partial \mathbf{r}}{\partial\theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} =$$

$$= R^2\sin\phi \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

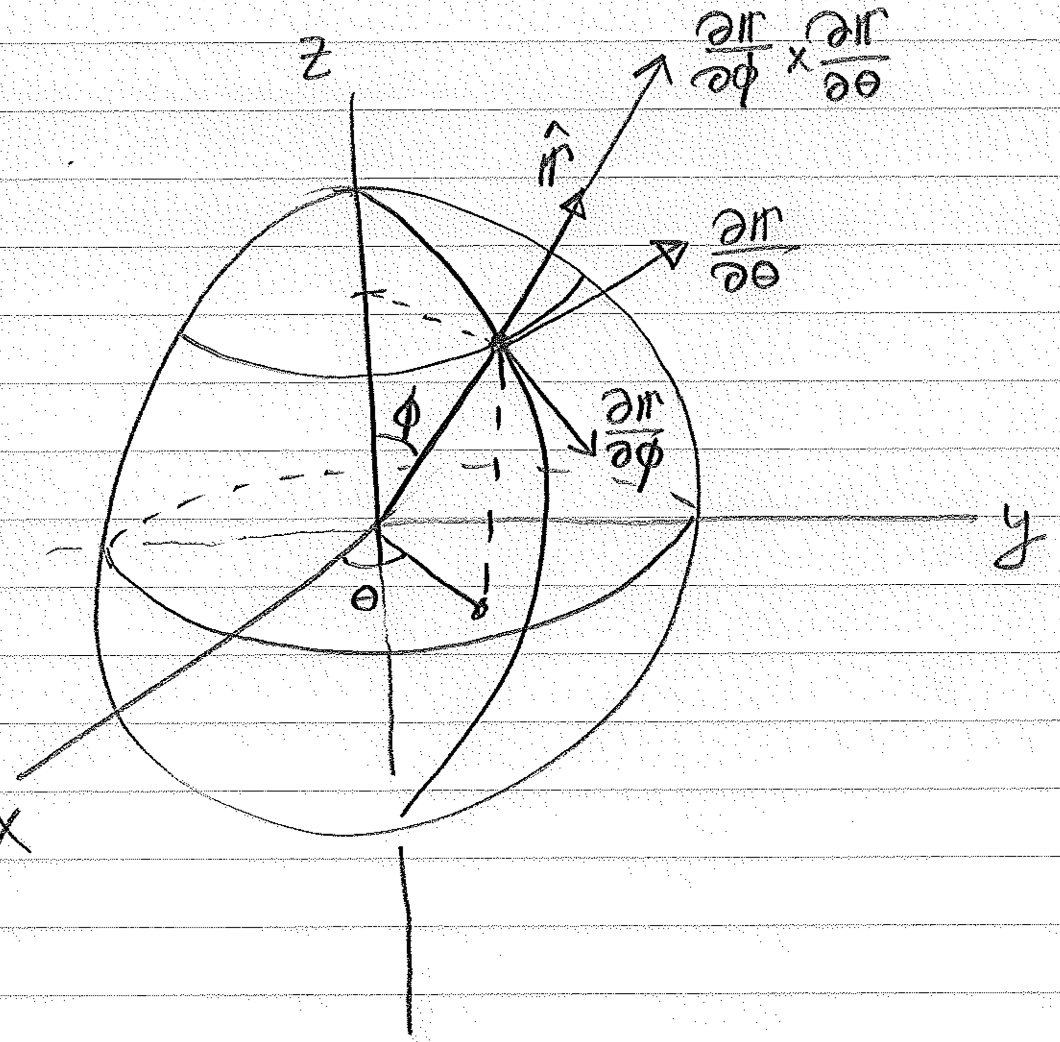
$$= R^2\sin\phi\,\underbrace{\left(\sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}\right)}_{=\frac{\mathbf{r}}{R}=\frac{\mathbf{r}}{|\mathbf{r}|}=\hat{\mathbf{r}}}$$

$$= R^2\sin\phi\,\hat{\mathbf{r}}$$

$$\left| \frac{\partial \mathbb{r}}{\partial \phi} \times \frac{\partial \mathbb{r}}{\partial \theta} \right| = R^2 \sin\phi \qquad \left(\sin\phi \geq 0 \underset{\text{for } \phi\in[0,\pi]}{} \right)$$

$$dS = R^2 \sin\phi \, d\phi \, d\theta \qquad \mathbb{N} = \pm R^2 \sin\phi \, \hat{\mathbb{r}}$$

$$d\mathbb{S} = \pm R^2 \sin\phi \, \hat{\mathbb{r}} \, d\phi \, d\theta \qquad \hat{\mathbb{N}} = \pm \hat{\mathbb{r}}$$



Sfärens area: $A = \iint_{S} dS = \iint_{D} R^2 \sin\phi \, d\phi \, d\theta$

$= \{\text{rektangel}\} = R^2 \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta =$

$= R^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = R^2 \cdot 2\pi \cdot 2 = 4\pi R^2$

Flödet av $\mathbb{F}(\mathbb{r}) = \dfrac{\mathbb{r}}{|\mathbb{r}|^3}$ <u>ut</u> genom $\mathcal{S}$:

$$\iint_{\mathcal{S}} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS = \iint_{D} \frac{\mathbb{r}}{|\mathbb{r}|^3} \cdot \underbrace{\left(\underset{\uparrow \text{ut}}{\overset{+}{(-)}}\, \hat{\mathbb{r}}\right)}_{=\hat{\mathbb{N}}} \underbrace{R^2 \sin\phi \; d\phi\, d\theta}_{dS}$$

$$= \iint_{D} \underbrace{\frac{\mathbb{r} \cdot \hat{\mathbb{r}}}{|\mathbb{r}|^3}}_{= \frac{R}{R^3}} \; R^2 \sin\phi \; d\phi \; d\theta$$

$$= \iint_{D} \sin\phi \; d\phi \; d\theta = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = 4\pi$$

Grad, div, rot 16.1

Med nabla-operatorn

$$\nabla = \frac{\partial}{\partial x}\mathbb{i} + \frac{\partial}{\partial y}\mathbb{j} + \frac{\partial}{\partial z}\mathbb{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

kan vi bilda

gradienten av skalärt fält:

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbb{i} + \frac{\partial\phi}{\partial y}\mathbb{j} + \frac{\partial\phi}{\partial z}\mathbb{k}$$

och divergensen och rotationen av vektorfält:

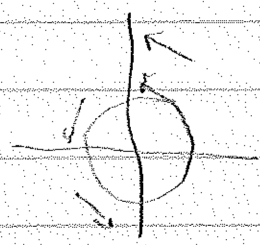
$$\nabla\cdot\mathbb{F} = \text{div}\,\mathbb{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(F_1, F_2, F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\times\mathbb{F} = \text{rot}\,\mathbb{F} = \text{curl}\,\mathbb{F} =$$

$$= \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Exempel Stelkroppsrotation



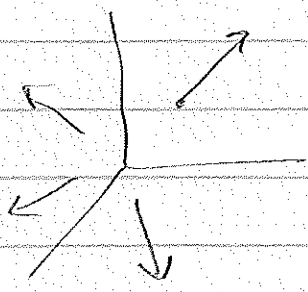
$$\mathbf{v} = \Omega(-y\mathbf{i} + x\mathbf{j})$$

$$\nabla \cdot \mathbf{v} = \Omega\left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x\right) = 0$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 2\Omega\mathbf{k}$$

Ingen divergens, bara rotation.

Exempel $\mathbf{F}(\mathbf{r}) = \mathbf{r}$



$$\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$$

$$\nabla \times \mathbf{F} = \ldots = \mathbf{0}$$

Divergerar, men roterar ej.

Exempel

$$\mathbb{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\mathbf{r}}{\rho^3} = \frac{(x,y,z)}{\rho^3}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) = \frac{\frac{\partial x}{\partial x}\rho^3 - x\frac{\partial \rho^3}{\partial x}}{\rho^3} =$$

$$= \frac{\rho^3 - x\cdot 3\rho^2\frac{\partial \rho}{\partial x}}{\rho^6} = \frac{\rho^3 - 3\frac{x^2}{\rho}}{\rho^6} = \frac{\rho^2 - 3x^2}{\rho^5}$$

På samma vis:

$$\frac{\partial F_2}{\partial y} = \frac{\rho^2 - 3y^2}{\rho^5}$$

$$\frac{\partial F_3}{\partial z} = \frac{\rho^2 - 3z^2}{\rho^5}$$

$$\boxed{\rho = \sqrt{x^2+y^2+z^2} \qquad \frac{\partial \rho}{\partial x} = \frac{2x}{2\sqrt{\cdots}} = \frac{x}{\rho}}$$

Så att $\nabla\cdot\mathbb{F} = \dfrac{3\rho^2 - 3(x^2+y^2+z^2)}{\rho^5} = 0, \ \rho \neq 0$

Divergensfritt utom i origo.

Vi har redan sett att flödet ut genom sfären är

$$\iint_S \mathbb{F}\cdot d\mathbf{S} = 4\pi$$

Vi ska återkomma till detta exempel senare.

Räkneregler 16.2

Sats 3

a) $\nabla(\phi\psi) = \nabla\phi\,\psi + \phi\nabla\psi$

b) $\nabla\cdot(\phi\mathbb{F}) = \nabla\phi\cdot\mathbb{F} + \phi\nabla\cdot\mathbb{F}$

c) d) e) f) i) läs boken

g) $\nabla\cdot(\nabla\times\mathbb{F}) = 0$

h) $\nabla\times\nabla\phi = \mathbb{O}$

Gör bevis av a, b, c, d, g, h
Resten utan bevis.

Obs: $\mathbb{F} = \nabla\phi \overset{(h)}{\Longrightarrow} \nabla\times\mathbb{F} = \mathbb{O}$

Alltså: ett konservativt fält är rotationsfritt. Detta är det nödvändiga villkoret i 15.2

Det är också tillräckligt i enkelt sammanhängande område enligt följande sats.

<u>Sats 4</u> Antag: 1) $\nabla \times \mathbb{F} = 0$ i D

2) D är enkelt sammanhängande.

Då är $\mathbb{F}$ konservativt i D, dvs det finns ϕ så att $\mathbb{F} = \nabla\phi$ i D.

Det betyder att ekv. systemet

$$\begin{cases} \dfrac{\partial\phi}{\partial x} = F_1 \\ \dfrac{\partial\phi}{\partial y} = F_2 \\ \dfrac{\partial\phi}{\partial z} = F_3 \end{cases}$$

är lösbart i D.

Exempel $\mathbf{F} = \mathbf{r}$. Då är $\nabla \times \mathbf{F} = \ldots = \mathbf{0}$.

Vi löser

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = x \\ \dfrac{\partial \phi}{\partial y} = y \\ \dfrac{\partial \phi}{\partial z} = z \end{cases} \Rightarrow \begin{cases} \phi(x,y,z) = \frac{1}{2}x^2 + f(y,z) \\ \phi(x,y,z) = \frac{1}{2}y^2 + g(x,z) \\ \phi(x,y,z) = \frac{1}{2}z^2 + h(x,y) \end{cases}$$

Välj
$$f(y,z) = \tfrac{1}{2}(y^2 + z^2) + C$$
$$g(x,z) = \tfrac{1}{2}(x^2 + z^2) + C$$
$$h(x,y) = \tfrac{1}{2}(x^2 + y^2) + C$$

Då fås

$$\phi(x,y,z) = \tfrac{1}{2}(x^2 + y^2 + z^2) + C.$$