RÖ12 GRAD, DIV, ROT

Nabla-operatorn $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

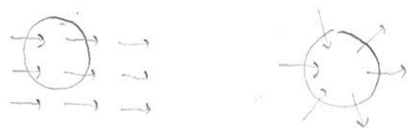
- Gradient $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, ger riktningen som funktionen ökar mest i

- Divergens $\text{div}\,\mathbb{F} = \nabla \cdot \mathbb{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, anger hur mkt som flödar in jämfört med hur mkt som flödar ut ur ett område

- Rotation $\text{rot}\,\mathbb{F} = \text{curl}\,\mathbb{F} = \nabla \times \mathbb{F}$, anger hur mycket fältet $\mathbb{F}$ roterar i en punkt.

Exempel divergens

Divergensfria fält



Lika mycket går in som ut

Divergerande fält, $\text{div}\,\mathbb{F} > 0$

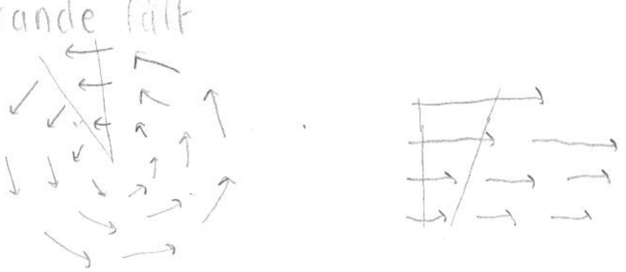
Mer går ut än in – måste finnas en källa i området/punkten.

"Konvergerande" fält, $\text{div}\,\mathbb{F} < 0$

Mer går in än ut – området/punkten är en sänka

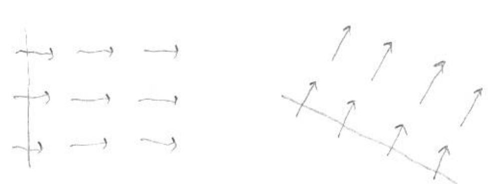
Exempel rotation

Roterande fält



om vi lägger en pinne vinkelrätt mot fältets/flödets riktn. kommer pinnen rotera

Rotationsfritt fält, $\nabla \times \mathbb{F} = 0$

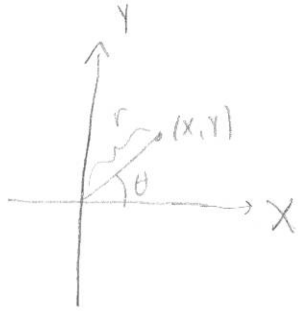


Pinnen kommer inte rotera

16.1.11. Beräkna div $\mathbb{F}$ och rot $\mathbb{F}$ då $\mathbb{F} = \hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$.

Notera att $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, men $\mathbb{F}$ beror inte direkt på x och y.

Kom ihåg att $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{matrix} \cos\theta = \frac{x}{r} \\ \sin\theta = \frac{y}{r} \end{matrix}$



$$r = \sqrt{x^2+y^2} \Rightarrow \begin{cases} \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \\ \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \end{cases}$$

Så div $\mathbb{F} = \nabla\cdot\mathbb{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial(-\sin\theta)}{\partial x} + \frac{\partial \cos\theta}{\partial y}$

$$= \frac{-\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= -\left(-\frac{1}{2}\right)\frac{\cdot 2xy}{(x^2+y^2)^{3/2}} + \left(-\frac{1}{2}\right)\frac{\cdot 2yx}{(x^2+y^2)^{3/2}}$$

$$= \frac{xy}{(x^2+y^2)^{3/2}} - \frac{xy}{(x^2+y^2)^{3/2}} = 0 \Rightarrow \mathbb{F} \text{ divergensfritt}$$

$$\text{rot }\mathbb{F} = \nabla\times\mathbb{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= \hat{\imath}\cdot 0 + \hat{\jmath}\cdot 0 + \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)\right)$$

$$= \hat{k}\left(\frac{\sqrt{x^2+y^2} - x\cdot\frac{1}{2}\cdot 2x(x^2+y^2)^{-1/2}}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - y\cdot\frac{1}{2}y\cdot(x^2+y^2)^{-1/2}}{x^2+y^2}\right)$$

$$= \hat{k}\left(\frac{2\sqrt{x^2+y^2} - (x^2+y^2)(x^2+y^2)^{-1/2}}{x^2+y^2}\right) = \hat{k}\cdot\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}\hat{k}$$

$\Rightarrow \mathbb{F}$ har rotation som avtar med avståndet från origo.

OBS att rotationen alltid blir runt z-axeln ($\hat{\mathbb{k}}$) då $\mathbb{F} = \mathbb{F}(\underset{\sim}{x, y})$, en fkn av x och y.

16.2.3. Visa att $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$.

$$VL: \nabla \cdot (F \times G) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \hat{\mathbb{i}} & \hat{\mathbb{j}} & \hat{\mathbb{k}} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(F_2 G_3 - F_3 G_2, F_3 G_1 - F_1 G_3, F_1 G_2 - F_2 G_1\right)$$

$$= \frac{\partial(F_2 G_3)}{\partial x} - \frac{\partial(F_3 G_2)}{\partial x} + \frac{\partial(F_3 G_1)}{\partial y} - \frac{\partial(F_1 G_3)}{\partial y} + \frac{\partial(F_1 G_2)}{\partial z} - \frac{\partial(F_2 G_1)}{\partial z}$$

$$= F_2 \frac{\partial G_3}{\partial x} + G_3 \frac{\partial F_2}{\partial x} - F_3 \frac{\partial G_2}{\partial x} - G_2 \frac{\partial F_3}{\partial x} + F_3 \frac{\partial G_1}{\partial y} + G_1 \frac{\partial F_3}{\partial y} - F_1 \frac{\partial G_3}{\partial y} - G_3 \frac{\partial F_1}{\partial y}$$

$$+ F_1 \frac{\partial G_2}{\partial z} + G_2 \frac{\partial F_1}{\partial z} - F_2 \frac{\partial G_1}{\partial z} - G_1 \frac{\partial F_2}{\partial z} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\cdot (G_1, G_2, G_3) - (F_1, F_2, F_3) \cdot \left(\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y}, \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}, \frac{\partial G_1}{\partial y} - \frac{\partial F_2}{\partial z}\right)$$

$$\underline{HL}: (\nabla \times F) \cdot G - F \cdot (\nabla \times G) = \begin{vmatrix} \hat{\mathbb{i}} & \hat{\mathbb{j}} & \hat{\mathbb{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot (G_1, G_2, G_3) - (F_1, F_2, F_3) \cdot \begin{vmatrix} \hat{\mathbb{i}} & \hat{\mathbb{j}} & \hat{\mathbb{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \cdot (G_1, G_2, G_3)$$

$$- (F_1, F_2, F_3) \cdot \left(\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y}, \frac{\partial G_3}{\partial x} \frac{\partial G_1}{\partial z}, \frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x}\right) = VL \quad VSV.$$