Idag: 4.5 Trippelintegralen

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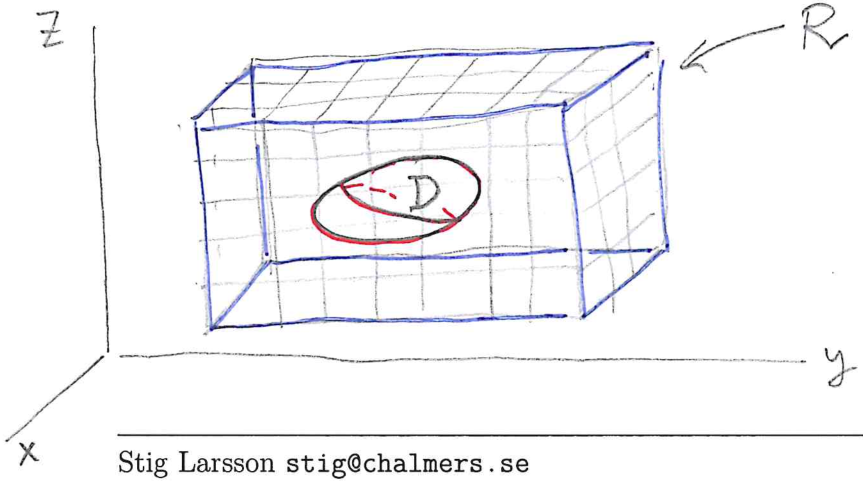
$$\iiint_R f\,dV = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

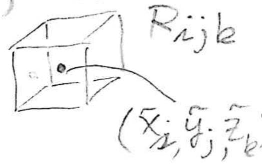
$f: \mathbb{R}^3 \to \mathbb{R}$

rätblock $R = [a,b]\times[c,d]\times[e,g]$



Partition $P = \{(x_i, y_j, z_k)\}_{i=0,\,j=0,\,k=1}^{n\quad m\quad p}$

$$R_{ijk} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]\times[z_{k-1}, z_k]$$



Riemann-summa

$$I(f,P) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(\tilde{x}_i, \tilde{y}_j, \tilde{z}_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

Om f kontinuerlig:

$$I(f,P) \longrightarrow \iiint_R f\,dV \text{ då } \max_{ijk} \operatorname{diam}(R_{ijk}) \to 0$$

Fubini:

$$\iiint_R f\,dV = \int_e^g \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz$$

$$= \int_c^d \left(\int_a^b \left(\int_e^g f(x,y,z)\,dz \right) dx \right) dy$$

$$= \int_a^b \left(\int_e^g \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx$$

Allmänt område D.

mätbart: vol(D) existerar

begränsat

$$f_D(x,y,z) = \begin{cases} f(x,y,z) & \text{i } D \\ 0 & \text{i } R\setminus D \end{cases}$$

$$\iiint_D f\,dV = \iiint_R f_D\,dV$$

Matlab!

Generaliserad integral om

D eller f obegränsade.

Sats 4.7 (Egenskaper)

$$\iiint_D 1\,dV = \mathrm{vol}(D)$$

$$\iiint_D f\,dV = 0 \quad \text{om } \mathrm{vol}(D) = 0$$

och så vidare.

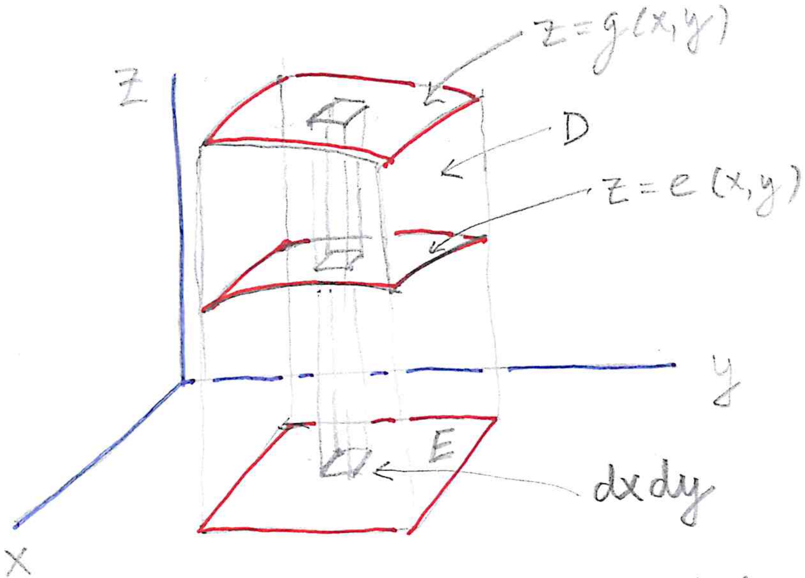
Upprepad integration

1. D enkelt i z

Mellan två grafer:

$z = e(x,y)$, $(x,y) \in E$

$z = g(x,y)$, $(x,y) \in E$



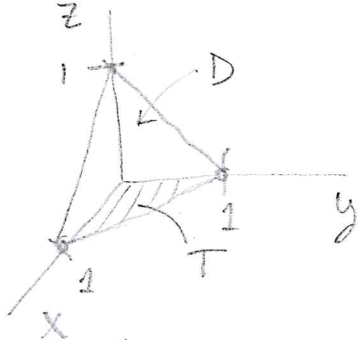
$$D = \{(x,y,z) : e(x,y) \le z \le g(x,y), (x,y) \in E\}$$

D mätbar med $\mathrm{vol}(D) = \iint_E (g(x,y) - e(x,y))\,dx\,dy$

$$\iiint_D f\,dV = \iint_E \left(\int_{e(x,y)}^{g(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

Dubbelintegral över E beräknas på lämpligt sätt. Rita figurer !!

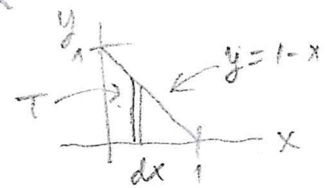
Ex Volymen av tetraeder.



Enkel i z: $0 \le z \le 1-x-y$

$$\mathrm{vol}(D) = \iint_T \left(\int_0^{1-x-y} dz \right) dx\,dy$$

$$= \iint_T (1-x-y)\,dx\,dy$$

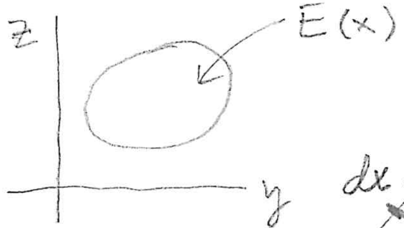


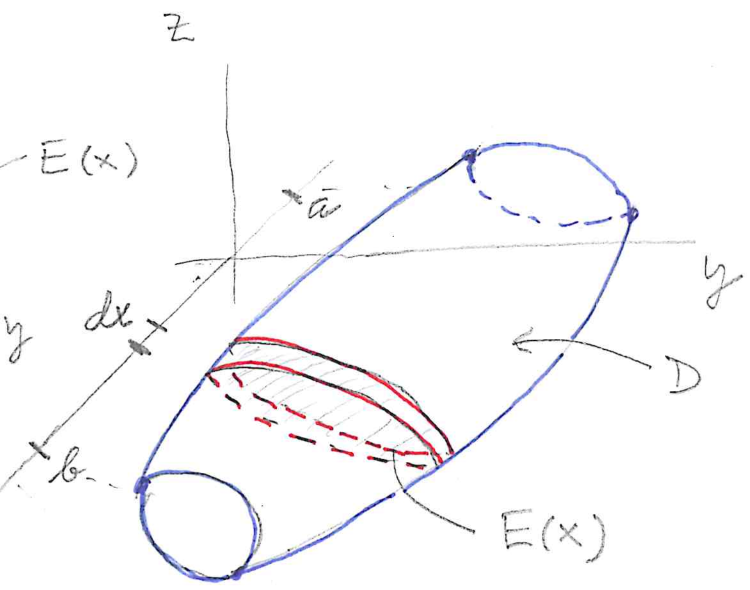
$= \{$ T är enkel i y $\} =$

$$= \int_0^1 \left(\int_0^{1-x} (1-x-y)\,dy \right) dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{1-x} dx = \dots = \frac{1}{6}$$

2. Skivningsmetoden





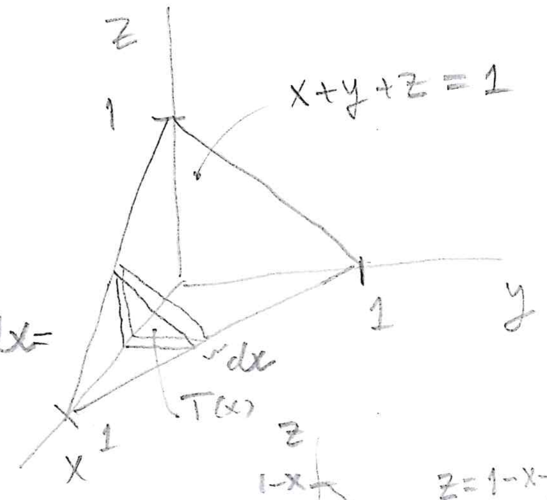
$$D = \{(x,y,z) : (y,z) \in E(x),\ a \le x \le b\}$$

$$\mathrm{vol}(D) = \int_a^b \mathrm{area}(E(x))\,dx$$

$$\iiint_D f\,dV = \int_a^b \left(\iint_{E(x)} f(x,y,z)\,dy\,dz \right) dx$$

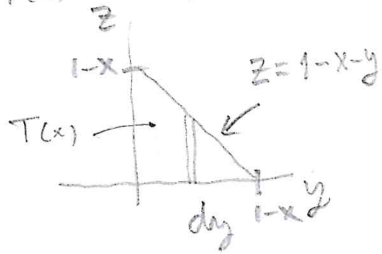
Ex Volym av tetraeder.

Skivning.



$$\mathrm{vol}(D) = \iiint_D dV =$$

$$= \int_0^1 \Big(\iint_{T(x)} dy\,dz \Big) dx =$$



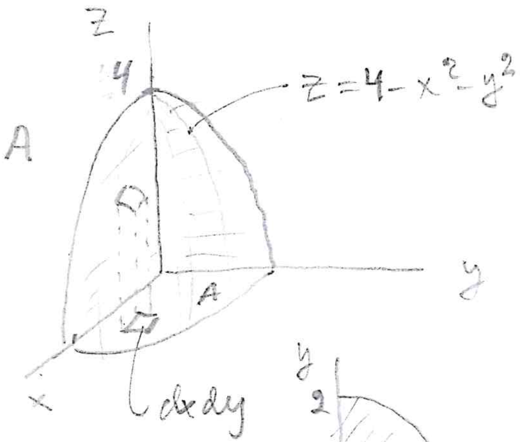
$$= \int_0^1 \Big(\int_0^{1-x} \Big(\int_0^{1-x-y} dz \Big) dy \Big) dx =$$

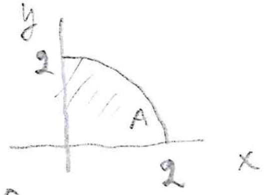
$$= \frac{1}{6}$$

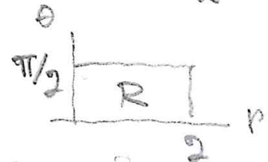
Ex $f(x,y,z)=x$, D under $z=4-x^2-y^2$ och i första oktanten.

Enkelt i z:

$$0 \le z \le 4-x^2-y^2, \ (x,y)\in A$$







$$\iiint_D x\,dx\,dy\,dz =$$

$$= \iint_A \Big(\int_0^{4-x^2-y^2} x\,dz \Big) dx\,dy$$

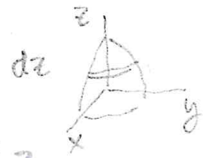
$$= \iint_A x \big[z \big]_{z=0}^{4-x^2-y^2} dx\,dy$$

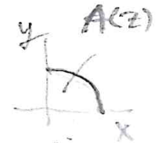
$$= \iint_A x(4-x^2-y^2)\,dx\,dy = \{\text{polära}\} =$$

$$= \iint_R r\cos\theta\,(4-r^2)\,r\,dr\,d\theta = \{\text{Fubini}\}$$

$$= \int_0^2 r^2(4-r^2)\,dr \int_0^{\pi/2} \cos\theta\,d\theta = \frac{64}{15}\cdot 1 = \frac{64}{15}$$

Skivning:





radie $= \sqrt{4-z}$

$$\iiint_D x\,dx\,dy\,dz = \int_0^2 \Big(\iint_{A(z)} x\,dx\,dy \Big) dz$$

$$= \int_0^2 \iint_{E(z)} r\cos\theta\,r\,dr\,d\theta = \int_0^2 \int_0^{\sqrt{4-z}} r^2\,dr \int_0^{\pi/2} \cos\theta\,d\theta\,dz.$$