Idag: Variabelsubstitution 4.6
Medelvärde. Moment. 4.7

4.6 Variabelsubstitution

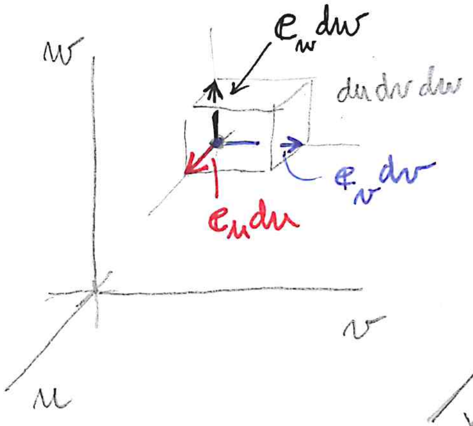
Transformation

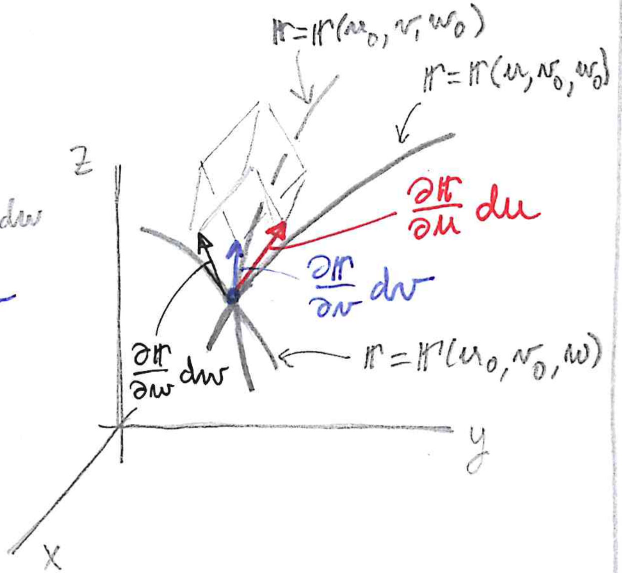
$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

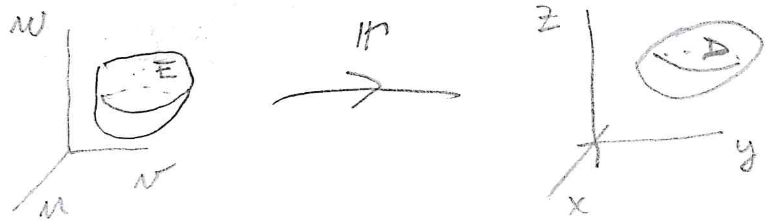
$\mathbf{r} = \mathbf{r}(u,v,w)$

Invers transformation

$$\begin{cases} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{cases}$$







Koordinatkurvor:

$\mathbf{r} = \mathbf{r}(u, v_0, w_0)$, $u \in \mathbb{R}$
$\mathbf{r} = \mathbf{r}(u_0, v, w_0)$, $v \in \mathbb{R}$
$\mathbf{r} = \mathbf{r}(u_0, v_0, w)$, $w \in \mathbb{R}$

Tangenter: $\frac{\partial \mathbf{r}}{\partial u}(u_0,v_0,w_0)$

$\frac{\partial \mathbf{r}}{\partial v}(u_0,v_0,w_0)$

$\frac{\partial \mathbf{r}}{\partial w}(u_0,v_0,w_0)$

dV spänns upp av tangenterna:

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u}du \cdot \left(\frac{\partial \mathbf{r}}{\partial v}dv \times \frac{\partial \mathbf{r}}{\partial w}dw\right)\right|$$

(beloppet av skalära trippelprodukten)

$$\frac{\partial \mathbf{r}}{\partial u}\cdot\left(\frac{\partial \mathbf{r}}{\partial v}\times\frac{\partial \mathbf{r}}{\partial w}\right) = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x'_w & y'_w & z'_w \end{vmatrix} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} =$$

$$= \det(\mathbf{r}'(u,v,w)) = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

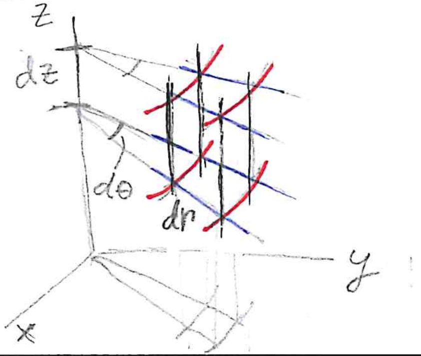
Volymskalan: $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = |\det(\mathbf{r}'(u,v,w))|$

beloppet av Jacobi-determinanten

Sats 4.8

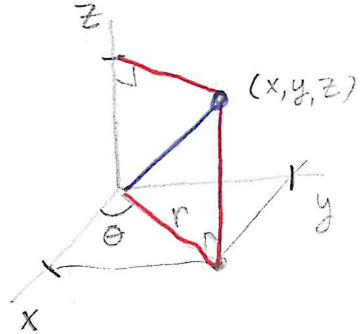
$$\iiint_D f(x,y,z)\,\underbrace{dx\,dy\,dz}_{dV(x,y,z)} =$$

$$= \iiint_E f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \underbrace{du\,dv\,dw}_{= dV(u,v,w)}$$



Cylindriska koordinater (r,θ,z)

$$\begin{cases} x = r\cos(\theta), \\ y = r\sin(\theta), \\ z = z, \end{cases} \qquad \begin{aligned} & r \in [0,\infty) \\ & \theta \in [0, 2\pi) \\ & z \in (-\infty,\infty) \end{aligned}$$



Invers:

$$\begin{cases} r = \sqrt{x^2+y^2+z^2}, \\ \theta = \operatorname{atan}(y/x), \\ z = z, \end{cases} \qquad \sqrt{x^2+y^2} \neq 0$$

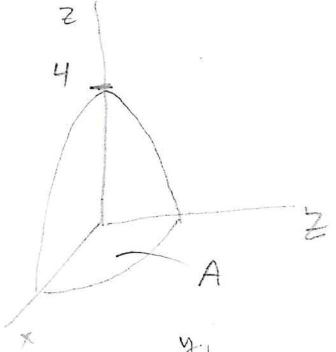
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x'_r & y'_r & z'_r \\ x'_\theta & y'_\theta & z'_\theta \\ x'_z & y'_z & z'_z \end{vmatrix} =$$

$$= \begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

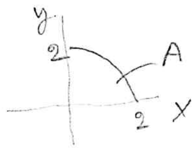
$$= r(\cos^2(\theta) + \sin^2(\theta)) = r$$

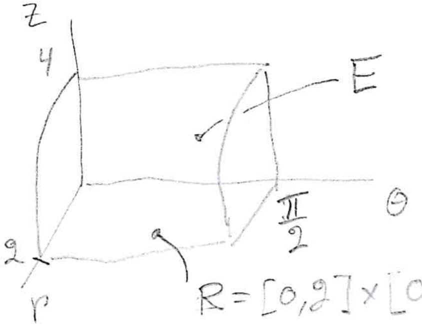
$$dV = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

Ex $f(x,y,z) = x$, D under $z = 4 - x^2 - y^2$ och i 1:a oktanten



Polära: $0 \le z \le 4 - r^2$
$0 \le \theta \le \frac{\pi}{2}$
$0 \le r \le 2$





$R = [0,2] \times [0, \frac{\pi}{2}]$

$$\iiint_D x \, dx\,dy\,dz = \iiint_E r\cos(\theta)\, r\, dr\, d\theta =$$

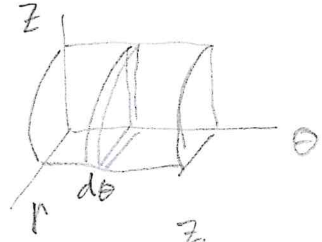
$$= \iiint_E r^2 \cos(\theta)\, dr\, d\theta = \{E \text{ enkel i } z\} =$$

$$= \iint_R \Big(\int_0^{4-r^2} r^2 \cos(\theta)\, dz \Big)\, dr\, d\theta$$

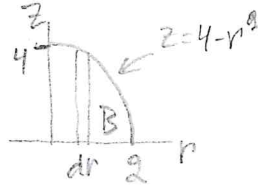
$$= \iint_R (4-r^2) r^2 \cos(\theta)\, dr\, d\theta = \{\text{Fubini}\}$$

$$= \int_0^2 (4-r^2) r^2\, dr \int_0^{\pi/2} \cos(\theta)\, d\theta = \ldots = \frac{64}{15}$$

Skivning:



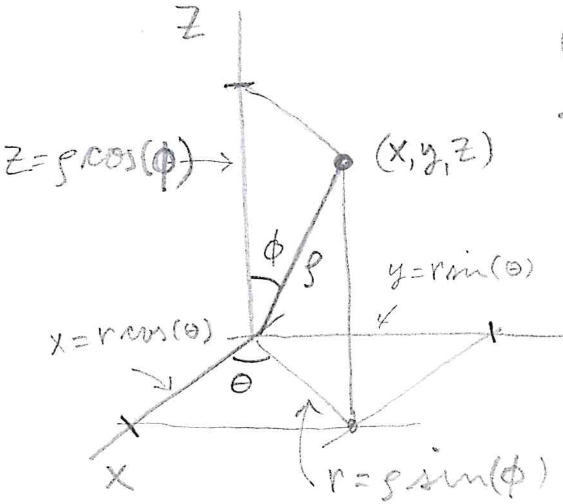
$$\int_0^{\pi/2} \Big(\iint_B r^2 \cos(\theta)\, dr\, dz \Big)\, d\theta =$$



$$= \int_0^{\pi/2} \Big(\int_0^2 \Big(\int_0^{4-r^2} r^2 \cos(\theta)\, dz \Big)\, dr \Big)\, d\theta =$$

$$= \int_0^{\pi/2} \Big(\int_0^2 \Big(r^2 \cos(\theta) \int_0^{4-r^2} dz \Big)\, dr \Big)\, d\theta = \ldots = \frac{64}{15}$$

Sfäriska koordinater (ρ, ϕ, θ)



$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases}$$

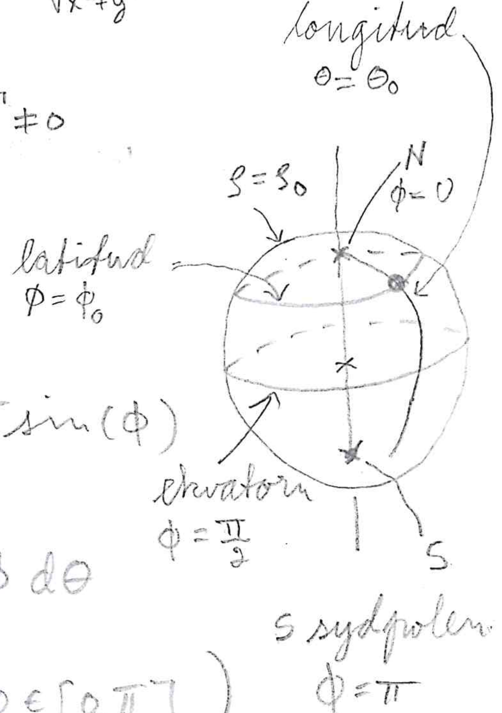
$\rho \in [0, \infty)$
$\phi \in [0, \pi]$
$\theta \in [0, 2\pi)$

$r = \sqrt{x^2 + y^2}$

$\cot(\phi)=\frac{z}{r}=\frac{z}{\sqrt{x^2+y^2}}$

Invers:

$$\begin{cases} \rho=\sqrt{x^2+y^2+z^2}, \\ \phi=\operatorname{acot}\left(\frac{z}{\sqrt{x^2+y^2}}\right), \\ \theta=\operatorname{atan}(y/x), \end{cases} \qquad \sqrt{x^2+y^2}\neq 0$$



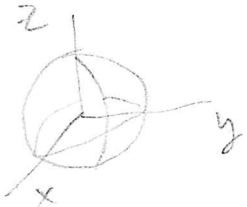
$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\ \cdots\ =\rho^2\sin(\phi)$$

$$dV=\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

(obs: $\sin(\phi)\geq 0$ då $\phi\in[0,\pi]$)

Ex $f(x,y,z)=x^2+y^2$,

$B:\ x^2+y^2+z^2\leq R^2$



Sfäriska: $E=[0,R]\times[0,\pi]\times[0,2\pi]$

$$\iiint_B (x^2+y^2)\,dx\,dy\,dz=\iiint_E (\rho\sin(\phi))^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\{\text{Fubini}\}=\int_0^R\rho^4\,d\rho\int_0^\pi\sin^2(\phi)\,d\phi\int_0^{2\pi}d\theta=\left\{\begin{aligned}&u=-\cos\phi\\&\sin^2(\phi)=\\&=(1-\cos^2\phi)\sin\phi\end{aligned}\right\}$$

$$=\frac{1}{5}R^5\cdot\int_{-1}^{1}(1-u^2)\,du\cdot 2\pi=\frac{1}{5}\cdot\frac{4}{3}\cdot 2\pi R^5=\frac{8\pi}{15}R^5$$

4.7 Medelvärde. Moment.

Def (Medelvärde med vikt)

$w>0$, kont., i D (viktfunktion)

$$\bar{f}=\frac{\iiint_D f\,w\,dV}{\iiint_D w\,dV}$$

Speciellt: om $w=$ konstant

$$\bar{f}=\frac{\iiint_D f\,dV}{\iiint_D 1\,dV}=\frac{1}{\operatorname{vol}(D)}\iiint_D f\,dV$$

Sats (Medelvärdessatsen)

D sammanhängande (ej två delar)
mätbar, sluten, begränsad
f kont. i D

Då $\exists\,(x_0,y_0,z_0)\in D$ så att

$$f(x_0,y_0,z_0)=\frac{\iiint_D f\,w\,dV}{\iiint_D w\,dV}$$

Ex Massa

d densitet $\left[\frac{kg}{m^3}\right]$

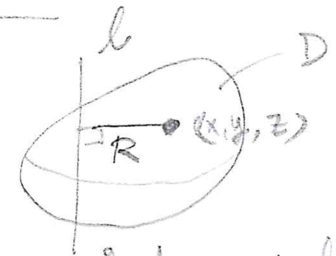
$$M=\iiint_D d\,dV=\iiint d(x,y,z)\,dx\,dy\,dz$$
$$=\iiint_D dm,\quad [kg]\quad dm=d\,dV$$

$$\bar{\mathbf{r}}=\frac{\iiint_D \mathbf{r}\,dm}{\iiint_D dm}=\frac{\iiint_D (x,y,z)\,dm}{M}\quad [m]$$

Medelvärdessatsen: $\exists\,\mathbf{r}_0$ så att $\mathbf{r}_0=\bar{\mathbf{r}}$. Tyngdpunkten. (masscentrum)

Ex Tröghetsmoment

$R(x,y,z)$ avstånd till axeln



$$I=\iiint_D R^2\,dm=\iiint_D R(x,y,z)^2\,d(x,y,z)\,dx\,dy\,dz$$

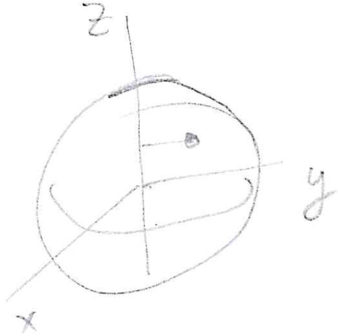
$[kg\,m^2]$

Med avs. på z-axeln:

$$I=\iiint_D (x^2+y^2)\,dm$$

Ex Tröghetsmoment för klot m.a.p. z-axeln.
Konstant densitet.

$$I = \iiint_B (x^2+y^2)\, d\, dV =$$



$$= \{\text{sfäriska}\}$$

$$= d \iiint_E (\rho \sin\phi)^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta =$$

$$= \{\text{som i förra exemplet}\} =$$

$$= d\,\frac{8\pi}{15} R^5 \quad [\text{kg m}^2]$$

Nästa vecka: integrera över <u>kurvor</u> och <u>ytor</u>!