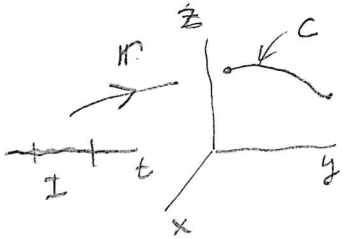
Idag: 5.1 Kurvor
5.2 Kurvintegraler

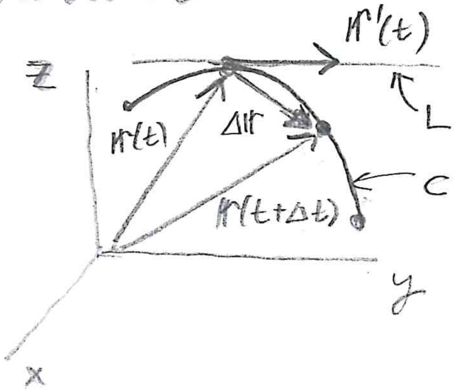
5.1 Kurva på parameterform

$C: \mathbf{r} = \mathbf{r}(t), \quad t \in I$



$C: \begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \quad t \in I$

$\mathbf{r}: \mathbb{R} \to \mathbb{R}^3$ kont. deriverbar



Medelhastighet:

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

Hastighet: $\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{r}'(t) = \dot{\mathbf{r}}(t)$

Fart: $v(t) = \|\mathbf{v}(t)\|$

Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

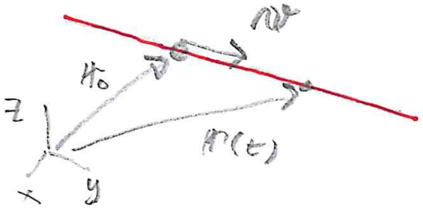
Om $\mathbf{r}'(t) \neq \mathbf{0}$, tangentvektor:

$$\mathbf{T}(t) = \mathbf{r}'(t)$$

Enhetstangent: $\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

C är slät om tangent finns.
(ej hörn, spets)

Ex rät linje



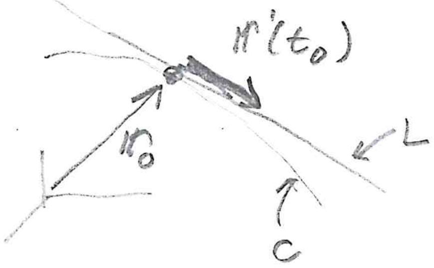
$L: \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

$\mathbf{r}'(t) = \mathbf{v}$

Ex tangentlinje

$C: \mathbf{r} = \mathbf{r}(t), \quad t \in I$

med $\mathbf{r}'(t_0) \neq \mathbf{0}$



Linjärisering:

$$\mathbb{L}_{t_0}(t) = \underbrace{\mathbf{r}(t_0)}_{=\mathbf{r}_0} + \mathbf{r}'(t_0)(t-t_0)$$

Rät linje:

$$L: \quad \mathbf{r} = \mathbf{r}_0 + (t-t_0)\mathbf{r}'(t_0) \qquad t \in \mathbb{R}$$

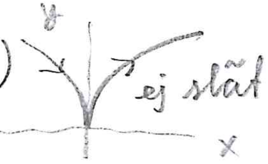
eller $(s = t - t_0)$

$$L: \quad \mathbf{r} = \mathbf{r}_0 + s\,\mathbf{r}'(t_0), \qquad s \in \mathbb{R}$$

Ex spets

$$C: \begin{cases} x = t^3 \\ y = t^2 \\ z = 0 \end{cases}, \quad t \in \mathbb{R}$$

$(t = x^{1/3}, y = x^{2/3})$



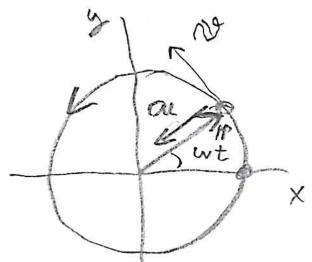
$\mathbf{r}(t) = (t^3, t^2, 0)$

$\mathbf{r}'(t) = (3t^2, 2t, 0), \quad \mathbf{r}'(0) = \mathbf{0}$

Graf: $y = x^{2/3}, \quad x \in \mathbb{R}. \qquad y' = \frac{2}{3}x^{-\frac{1}{3}}$

Ex cirkel

$R > 0, \quad \omega > 0$



$$C: \begin{cases} x = R\cos(\omega t) \\ y = R\sin(\omega t) \\ z = 0 \end{cases}, \quad t \in \mathbb{R}$$

$\mathbf{r} = (R\cos(\omega t), R\sin(\omega t), 0)$

$\mathbf{v} = (-R\omega\sin(\omega t), R\omega\cos(\omega t), 0)$

$v = \|\mathbf{v}\| = R\omega$

$\mathbf{v} \cdot \mathbf{r} = 0$

$\mathbf{a} = (-R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t), 0) = -\omega^2\mathbf{r}$

Ex olika parametrisering

halvcirkel, radie $= 1$

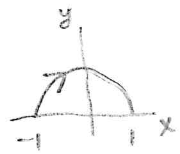
(a) välj $t = \theta$

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = 0 \end{cases}, \quad t \in [0, \pi]$$

$\mathbf{r}'(t) = (-\sin(t), \cos(t), 0), \quad \|\mathbf{r}'(t)\| = 1$

(b) välj $u = x$

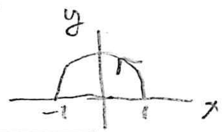
$$\begin{cases} x = u \\ y = \sqrt{1-u^2} \\ z = 0 \end{cases}, \quad u \in [-1, 1]$$



$\mathbf{r}'(u) = \left(1, -\frac{u}{\sqrt{1-u^2}}, 0\right), \quad \|\mathbf{r}'(u)\| = \frac{1}{\sqrt{1-u^2}}$

(c) välj $v = 1 - x$

$$\begin{cases} x = 1 - v \\ y = \sqrt{1-(1-v)^2} \\ z = 0 \end{cases}, \quad v \in [0, 2]$$



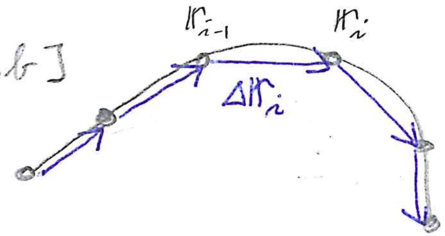
Kedjeregeln: $\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\underbrace{\dfrac{ds}{dt}}_{\text{skalär}}$

Ändrar fart, byter riktning om $\dfrac{ds}{dt}<0$.

5.2 Kurvintegraler

$C:\ \mathbf{r}=\mathbf{r}(t),\quad t\in[a,b]$

Längd L?



Partition: $a=t_0<\dots<t_i\le\dots<t_n=b$

Linjärisering:

$$\Delta\mathbf{r}_i = \mathbf{r}(t_i)-\mathbf{r}(t_{i-1}) = \mathbf{r}'(t_{i-1})\Delta t_i + \text{felet}$$

$$\Delta\mathbf{r}_i \approx \mathbf{r}'(t_{i-1})\Delta t_i$$

$$L \approx \sum_{i=1}^{n}\|\Delta\mathbf{r}_i\| \approx \sum_{i=1}^{n}\|\mathbf{r}'(t_{i-1})\|\Delta t_i$$

Riemann-summa för enkelintegral:

$$L = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_C ds$$

$ds = \|\mathbf{r}'(t)\|\,dt$ båglängdselement

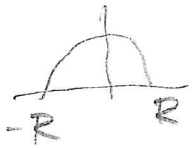
Kurvintegral:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt$$

Beror ej på val av parametrisering

Ex Halvcirkel, radie R

$f(x,y,z)=y$



(a) $\begin{cases} x=R\cos(t) \\ y=R\sin(t),\quad t\in[0,\pi] \\ z=0 \end{cases}$

$\mathbf{r}'(t)=(-R\sin(t), R\cos(t), 0),\quad \|\mathbf{r}'(t)\|=R$

$ds = R\,dt$

$$\int_C f\,ds = \int_0^{\pi}(R\sin(t))\,R\,dt = 2R^2$$

(b) $\begin{cases} x=u \\ y=\sqrt{R^2-u^2},\quad u\in[-R,R] \\ z=0 \end{cases}$

$\|\mathbf{r}'(u)\| = \dfrac{R}{\sqrt{R^2-u^2}}$

$$\int_C f\,ds = \int_{-R}^{R}\sqrt{R^2-u^2}\,\frac{R}{\sqrt{R^2-u^2}}\,du = 2R^2$$

Vektorbåglängdselement:

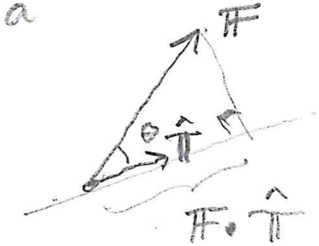
$$\Delta \mathbf{r}_i \approx \mathbf{r}'(t_{i-1})\,\Delta t_i$$

$$d\mathbf{r} = \mathbf{r}'(t)\,dt = \underbrace{\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}}_{=\hat{\mathbf{T}}} \underbrace{\|\mathbf{r}'(t)\|\,dt}_{=ds}$$

$$d\mathbf{r} = \hat{\mathbf{T}}\,ds = \mathbf{r}'(t)\,dt$$

Tangentkurvintegral:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\hat{\mathbf{T}}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$



$$\mathbf{F}\cdot\hat{\mathbf{T}} = \|\mathbf{F}\|\cos(\theta)$$

tangentkomponenten av $\mathbf{F}$

Ex arbete $\mathbf{F}$ [N] kraftfält

$$W = \int_C \mathbf{F}\cdot\hat{\mathbf{T}}\,ds \quad [\text{Nm}]$$

Konservativt fält i D om $\exists\,\phi$ så att

$$\mathbf{F} = \nabla\phi \text{ i } D$$

ϕ kallas potential

Ex tyngdkraftfältet $\mathbf{F} = -mg\,\mathbf{e}_z$ $\downarrow mg$

ϕ ?

$$\begin{cases} \phi'_x = 0 \\ \phi'_y = 0 \\ \phi'_z = -mg \end{cases} \Rightarrow \begin{cases} \phi(x,y,z) = f(y,z) \\ \phi(x,y,z) = g(x,z) \\ \phi(x,y,z) = -mgz + h(x,y) \end{cases}$$

$$\Rightarrow \phi(x,y,z) = -mgz + C \quad [\text{Nm}]$$

Kallas potentiell energi.

Sats: $\mathbf{F} = \nabla\phi$, D öppen, sammanhängande

$$\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$

oberoende av vägen

Bevis:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \nabla\phi(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b \frac{d}{dt}\phi(\mathbf{r}(t))\,dt$$

$$= \Big[\phi(\mathbf{r}(t))\Big]_a^b = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$

Konservativa fält är de enda fält vars integral är ober. av vägen enligt följande sats:

Sats D öppen, sammanhängande. Följande är ekvivalenta.

(a) $\mathbb{F}$ konservativt i D

(b) $\int_C \mathbb{F} \cdot d\mathbb{r} = 0$ för alla slutna kurvor i D

(c) $\int_C \mathbb{F} \cdot d\mathbb{r}$ är ober. av vägen

Har redan visat (a) $\Rightarrow$ (c).

Bevisidé: (b) $\Leftrightarrow$ (c)

$C = C_1 \cup C_2$

Bevisidé: (c) $\Rightarrow$ (a)

$$\phi(x,y,z) = \int_{P_0}^{(x,y,z)} \mathbb{F} \cdot d\mathbb{r}$$

Imorgon: ytor, ytintegraler.