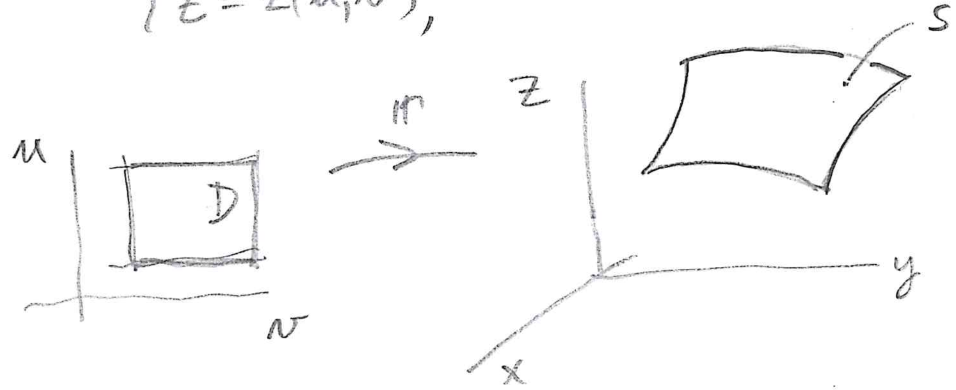
Idag: 5.3 Ytor
5.4 Ytintegraler

Yta på parameterform

$S: \mathbf{r} = \mathbf{r}(u,v), \quad (u,v) \in D$

$$S: \begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} \quad (u,v) \in D$$



$\mathbf{r}: \mathbb{R}^2 \to \mathbb{R}$, kontinuerligt deriverbar

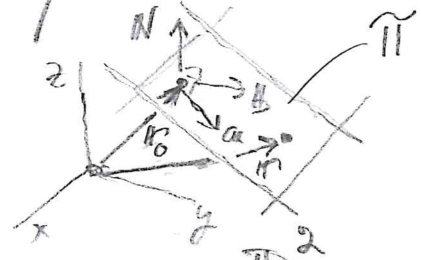
Ex planet (enklaste ytan)

P_0 punkt i planet

$\mathbf{a}, \mathbf{b}$ parallella med planet

ej kolinjära, dvs

$\mathbf{N} = \mathbf{a} \times \mathbf{b} \neq \mathbf{0}$

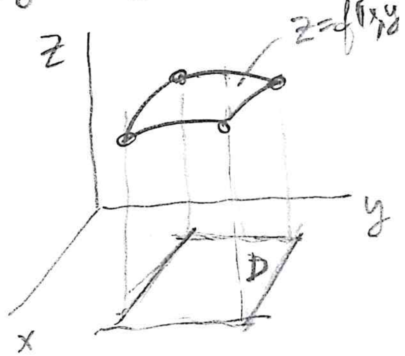


$\Pi: \mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}, \quad (u,v) \in \mathbb{R}^2$

Ex graf $z = f(x,y), \quad (x,y) \in D$

Välj $u = x, \; v = y$:

$$S: \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases} \quad (u,v) \in D$$

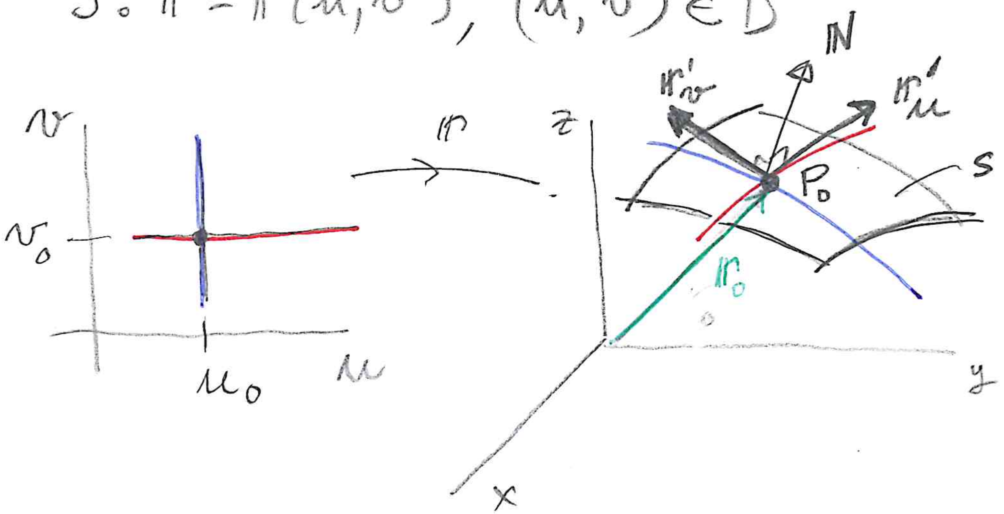


Ex sfär $x^2+y^2+z^2=R^2$, $\rho = R$, $u = \phi$, $v = \theta$

$$S: \begin{cases} x = R\sin(u)\cos(v) \\ y = R\sin(u)\sin(v) \\ z = R\cos(u) \end{cases}, \quad (u,v) \in [0,\pi]\times[0,2\pi]$$

Ex tangentplan

$S: \mathbf{r} = \mathbf{r}(u,v),\ (u,v) \in D$



Koordinatkurvor genom P_0, $\mathbf{r}_0 = \mathbf{r}(u_0,v_0)$

$$\mathbf{r} = \mathbf{r}(u, v_0),\quad u \in I_1$$
$$\mathbf{r} = \mathbf{r}(u_0, v),\quad v \in I_2$$

Tangenter i P_0:

$$\mathbf{r}'_u(u_0,v_0) = \frac{\partial \mathbf{r}}{\partial u}(u_0,v_0)$$
$$\mathbf{r}'_v(u_0,v_0) = \frac{\partial \mathbf{r}}{\partial v}(u_0,v_0)$$

Antag

$$\mathbf{N} = \mathbf{r}'_u(u_0,v_0) \times \mathbf{r}'_v(u_0,v_0) \neq \mathbf{0}$$

Tangenterna spänner upp tangentplanet i P_0

$$\Pi: \mathbf{r} = \mathbf{r}_0 + u\,\mathbf{r}'_u(u_0,v_0) + v\,\mathbf{r}'_v(u_0,v_0)$$
$$(u,v) \in \mathbb{R}^2$$

Enhetsnormalvektor:

$$\hat{\mathbf{N}} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \pm \frac{\mathbf{r}'_u \times \mathbf{r}'_v}{\|\mathbf{r}'_u \times \mathbf{r}'_v\|}$$

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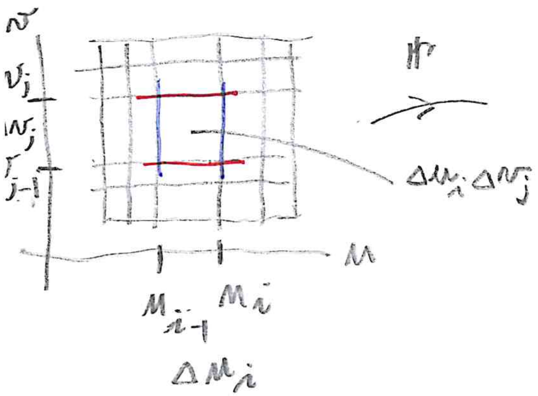
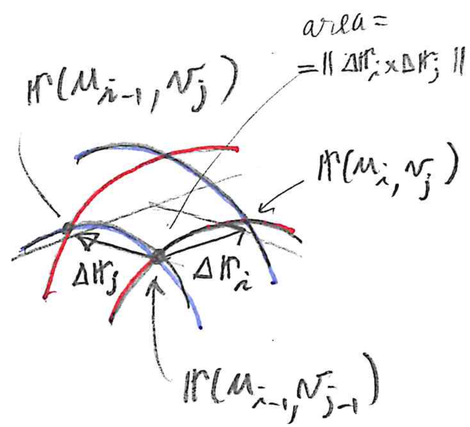
Ytintegraler

Bestäm area(S).

Partition av D





$\Delta \mathbf{r}_i \approx \mathbf{r}'_u(u_{i-1}, v_{j-1})\,\Delta u_i$ (linjärisering)

$\Delta \mathbf{r}_j \approx \mathbf{r}'_v(u_{i-1}, v_{j-1})\,\Delta v_j$

Area: $\|\Delta \mathbf{r}_i \times \Delta \mathbf{r}_j\| \approx$

$$\approx \|\mathbf{r}'_u(u_{i-1}, v_{j-1}) \times \mathbf{r}'_v(u_{i-1}, v_{j-1})\|\,\Delta u_i \Delta v_j$$

$$A \approx \sum_i \sum_j \|\mathbf{r}'_u(u_{i-1}, v_{j-1}) \times \mathbf{r}'_v(u_{i-1}, v_{j-1})\|\,\Delta u_i \Delta v_j$$

Riemann-summa för integralen

$$A = \iint_D \|\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\|\,du\,dv$$

Areaelement:

$$dS = \|\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\|\,du\,dv$$

så att

$$A = \iint_S dS$$

Ytintegralen av f över S:

$$\iint_S f\,dS =$$

$$= \iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\|\,du\,dv$$

Ex. graf $z = f(x,y)$, $(x,y) \in D$

$$S: \begin{cases} x = u, \\ y = v, \\ z = f(u,v), \end{cases} \quad (u,v) \in D$$

Tangenter: $\mathbf{r}'_u = (1, 0, f'_1(u,v))$, $\mathbf{r}'_v = (0, 1, f'_2(u,v))$

En normalvektor:

$$\mathbf{N} = \mathbf{r}'_u \times \mathbf{r}'_v = \ldots = (-f'_1(u,v), -f'_2(u,v), 1)$$

Areaelementet:

$$dS = \sqrt{1 + f_1'(u,v)^2 + f_2'(u,v)^2}\, du\, dv$$

eller med x, y:

$$dS = \sqrt{1 + f_x'(x,y)^2 + f_y'(x,y)^2}\, dx\, dy$$

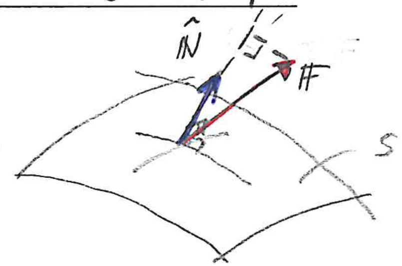
S kallas <u>orienterbar</u> om det finns normalvektorfält som varierar kontinuerligt över S.

Välj då ett sådant normalvektorfält $\Rightarrow$ <u>orienterad yta</u>

Dvs bestäm vad som är "upp/ned" eller "in/ut".

Möbius band är ej orienterbar.

S orienterad med normalvektorfält $\mathbb{N}$.

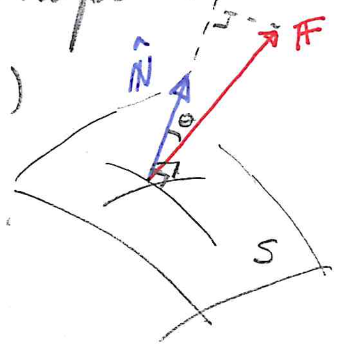


Bilda flödesintegralen av $\mathbb{F}$:

$$\iint_S \mathbb{F} \bullet \hat{\mathbb{N}}\, dS$$

Integrera normalkomponenten

$$\mathbb{F} \bullet \hat{\mathbb{N}} = \|\mathbb{F}\| \cos(\theta)$$



$$\hat{\mathbb{N}} = \pm \frac{\mathbb{r}_u' \times \mathbb{r}_v'}{\|\mathbb{r}_u' \times \mathbb{r}_v'\|}$$

val av orientering

Normalytelementet:

$$d\mathbb{S} = \hat{\mathbb{N}}\, dS = \pm \frac{\mathbb{r}_u' \times \mathbb{r}_v'}{\|\mathbb{r}_u' \times \mathbb{r}_v'\|} \|\mathbb{r}_u' \times \mathbb{r}_v'\|\, du\, dv$$

$$= \pm (\mathbb{r}_u' \times \mathbb{r}_v')\, du\, dv$$

Ex vätskeflöde

Hastighetsfält: $\mathbf{v}(x,y,z)$ $[m/s]$

Volymsflöde genom S:

$$\iint_S \mathbf{v}\cdot\hat{\mathbf{N}}\,dS \qquad [m^3/s]$$

Densitet d $[kg/m^3]$,

Massflöde genom S:

$$\iint_S d\,\mathbf{v}\cdot\hat{\mathbf{N}}\,dS \qquad [kg/s]$$

Ex sfären $x^2+y^2+z^2=R^2$

$$S:\begin{cases} x = R\sin(\phi)\cos(\theta) \\ y = R\sin(\phi)\sin(\theta) \\ z = R\cos(\phi) \end{cases} \qquad (\phi,\theta)\in[0,\pi]\times[0,2\pi]$$

Tangenter: $\mathbf{r}'_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi))$

$\mathbf{r}'_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0)$

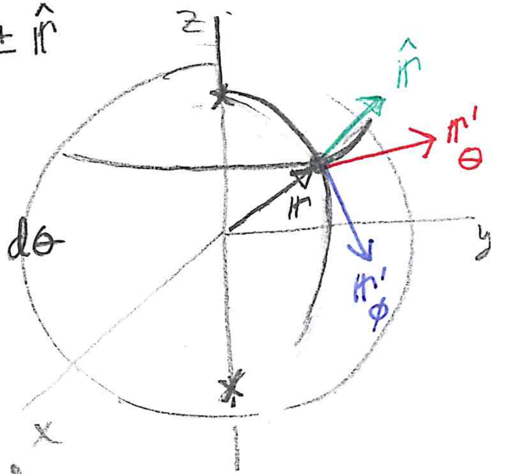
En normalvektor:

$$\mathbf{r}'_\phi\times\mathbf{r}'_\theta = \dots =$$

$$= R^2\sin(\phi)\underbrace{(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))}_{=\frac{\mathbf{r}}{R}=\frac{\mathbf{r}}{\|\mathbf{r}\|}=\hat{\mathbf{r}}}$$

$$= R^2\sin(\phi)\,\hat{\mathbf{r}}$$

$\|\mathbf{r}'_\phi\times\mathbf{r}'_\theta\| = R^2\sin(\phi)$, $\hat{\mathbf{N}} = \pm\hat{\mathbf{r}}$



$$dS = R^2\sin(\phi)\,d\phi\,d\theta$$

$$d\mathbf{S} = \hat{\mathbf{N}}\,dS = \underset{\substack{\uparrow \\ \text{orientering} \\ \text{in eller ut}}}{\pm}\hat{\mathbf{r}}\,R^2\sin(\phi)\,d\phi\,d\theta$$

$$A = \iint_S dS = \iint_D R^2\sin(\phi)\,d\phi\,d\theta = R^2\int_0^\pi \sin(\phi)\,d\phi\int_0^{2\pi}d\theta = 4\pi R^2$$

Flödet av $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ ut genom S:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \iint_D \underbrace{\frac{\mathbf{r}}{\|\mathbf{r}\|^3}\cdot\left(\substack{+\\(-)}\hat{\mathbf{r}}\right)}_{=1/R^2 \text{ på } S} R^2\sin(\phi)\,d\phi\,d\theta$$

$$= \iint_D \sin(\phi)\,d\phi\,d\theta = 4\pi$$