Idag: Taylors formel 1.6
Gradient 1.7

Taylors formel av ordning 2.

En variabel:

$$F(t) = F(\bar{t}) + F'(\bar{t})(t-\bar{t}) + \tfrac{1}{2}F''(\bar{t})(t-\bar{t})^2 + E_2[F,\bar{t}](t)$$

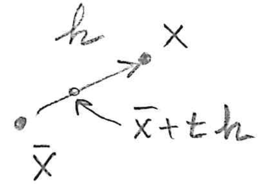
med $E_2[F,\bar{t}](t) = \frac{1}{6}F'''(s)(t-\bar{t})^3$,

där s är mellan t och $\bar{t}$.

Taylors polynom, grad 2, i $\bar{t}$:

$$P_2[F,\bar{t}](t) = F(\bar{t}) + F'(\bar{t})(t-\bar{t}) + \tfrac{1}{2}F''(\bar{t})(t-\bar{t})^2$$

Nu: $f: \mathbb{R}^2 \to \mathbb{R}$.



Bilda:

$$F(t) = f(\bar{x}+th), \quad F: \mathbb{R} \to \mathbb{R}$$

$$F(0) = f(\bar{x})$$

$$F(1) = f(x)$$

Taylor:

$$F(1) = F(0) + F'(0)\cdot 1 + \tfrac{1}{2}F''(0)\cdot 1^2 + \tfrac{1}{6}F'''(s)1^3.$$

Kedjeregeln: $\frac{d}{dt}(\bar{x}+th) = h$

$$F'(t) = f'(\bar{x}+th)h = f_1'(x+th)h_1 + f_2'(x+th)h_2$$

$$F'(0) = f'(\bar{x})h = \begin{bmatrix} f_1'(\bar{x}) & f_2'(\bar{x}) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = f_1'(\bar{x})h_1 + f_2'(\bar{x})h_2$$

$$F''(t) = \frac{d}{dt}\left(f_1'(\bar{x}+th)h_1 + f_2'(\bar{x}+th)h_2 \right)$$

$$= f_{11}''(x+th)h_1^2 + f_{12}''(\bar{x}+th)h_1h_2 +$$

$$+ f_{21}''(\bar{x}+th)h_2h_1 + f_{22}''(\bar{x}+th)h_2^2$$

$$F''(0) = f_{11}''(\bar{x})h_1^2 + 2f_{12}''(\bar{x})h_1h_2 + f_{22}''(x)h_2^2$$

$$= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} f_{11}''(\bar{x}) & f_{12}''(\bar{x}) \\ f_{12}''(\bar{x}) & f_{22}''(\bar{x}) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= h^T f''(\bar{x}) h \quad \text{(kvadratisk form)}$$

Def (Hesse-matrisen)

$f : \mathbb{R}^n \to \mathbb{R}$, med kontinuerliga part. derivator i omgivning av $\bar{x}$. Hessematrisen är

$$D^2f(\bar{x}) = f''(\bar{x}) = \left(f_{ij}''(\bar{x}) \right)_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$$

Den är symmetrisk, ty $\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Kan beräknas som

$$D^2 f(\bar{x}) = D(Df^T)(x)$$

dvs $D \begin{bmatrix} f_1' \\ \vdots \\ f_n' \end{bmatrix}$

Matlab:
```
>> f = @(x)( x(1)*x(2)*x(3))
>> Df = @(x) jacobi(f,x)'
>> D2f(x) = jacobi(Df, x)
```

Taylors poly. av grad 2 i x:

$$P_2[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})h + \frac{1}{2}h^T f''(\bar{x})h.$$

Resttermen:

$$E_2[f,\bar{x}](x) = \frac{1}{6}F'''(s) =$$

$$= \frac{1}{6}\Big(f'''_{111}(\xi)h_1^3 + 3f'''_{112}(\xi)h_1^2h_2 + 3f'''_{122}(\xi)h_1h_2^2 + f'''_{222}(\xi)h_2^3 \Big)$$

med $\xi = \bar{x} + sh$, $s \in [0,1]$, okänd punkt.

$$|E_2[f,\bar{x}](x)| \le K\|h\|^3 \quad \forall x \in B_\delta(\bar{x})$$

$$K \le c \max_{\xi \in B_\delta(\bar{x})} |f'''_{ijk}(\xi)|$$

Vi har bevisat Taylors formel av ordning 2 för $n=2$.

Sats 1.10 (Taylors sats)

Antag: $f : \mathbb{R}^n \to \mathbb{R}$ med kont. partiella derivator i omgivn $B_\delta(\bar{x})$. Då gäller

$$f(x) = P_2[f,\bar{x}](x) + E_2[f,\bar{x}](x)$$

med, $h = x - \bar{x}$, och

$$P_2[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})h + \frac{1}{2}h^T f''(\bar{x})h$$

$$|E_2[f,\bar{x}](x)| \le c_n \max_{\substack{\xi \in B_\delta(\bar{x}) \\ i,j,k}} |f'''_{ijk}(\xi)| \, \|h\|^3$$

Ex $f(x) = \frac{x_1}{x_2} + \frac{8}{x_1} - x_2$, $\bar{x} = (1,1)$, $f(1,1) = 8$

$$f'(x) = \left[\frac{1}{x_2} - \frac{8}{x_1^2}, \frac{-x_1}{x_2^2} - 1\right], \quad f'(1,1) = [-7, -2]$$

$$f''(x) = D\begin{bmatrix} \frac{1}{x_2} - \frac{8}{x_1^2} \\ \frac{-x_1}{x_2^2} - 1 \end{bmatrix} = \begin{bmatrix} \frac{16}{x_1^3} & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}, \quad f''(1,1) = \begin{bmatrix} 16 & -1 \\ -1 & 2 \end{bmatrix}$$

$$P_2[f,\bar{x}](x) = 8 + [-7, 2]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2}[h_1, h_2]\begin{bmatrix} 16 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

1.7 Gradient

Geometriska beteckningar i detta avsnitt!

$f: \mathbb{R}^3 \to \mathbb{R}$

$w = f(x,y,z)$ skalärt fält

$w = f(\mathbf{r})$, $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = (x,y,z)$

rumskoordinater

Gradienten till f:

$$\nabla f(x,y,z) = f'_x(x,y,z)\mathbf{e}_x + f'_y(x,y,z)\mathbf{e}_y + f'_z(x,y,z)\mathbf{e}_z$$
$$= \left(f'_x(x,y,z), f'_y(x,y,z), f'_z(x,y,z)\right)$$

$\nabla f: \mathbb{R}^3 \to \mathbb{R}^3$ vektorfält

Def 1.14 (Nabla-operatorn)

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Deriveringsoperator, verkar på skalärt fält: $\nabla f = (f'_x, f'_y, f'_z)$

Ex $f(x,y,z) = x^2+y^2+z^2$

$f(\mathbf{r}) = |\mathbf{r}|^2$

$\nabla f(x,y,z) = (2x, 2y, 2z) = 2\mathbf{r}$

$\nabla f(1,2,3) = (2,4,6) = 2(1,2,3)$

Def 1.15 (Riktningsderivata)

Låt $\mathbf{u} = (u_x, u_y, u_z)$ vara enhetsvektor, $|\mathbf{u}| = 1$.

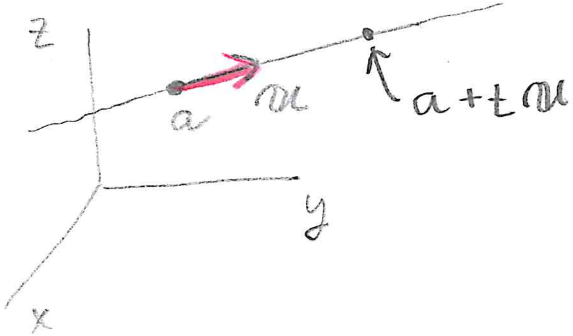
$$D_{\mathbf{u}} f(a) = \frac{d}{dt} f(a + t\mathbf{u})\Big|_{t=0}$$

Dvs derivera en-variabel-funktionen

$$g(t) = f(a+t\mathbf{u})$$

i $t=0$:

$$D_{\mathbf{u}}f(a) = g'(0)$$



Sats 1.11 (Riktningsderivata)

$$D_{\mathbf{u}}f(a) = \mathbf{u}\cdot\nabla f(a)$$

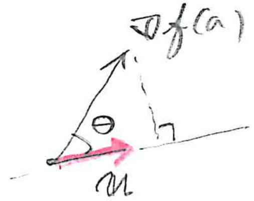
Bevis Kedjeregeln:

$$\frac{d}{dt}f(a+t\mathbf{u}) = f'_x(a+t\mathbf{u})u_x + f'_y(a+t\mathbf{u})u_y + f'_z(a+t\mathbf{u})u_z$$

Med $t=0$:

$$D_{\mathbf{u}}f(a) = f'_x(a)u_x + f'_y(a)u_y + f'_z(a)u_z = \mathbf{u}\cdot\nabla f(a).$$

Obs: $\mathbf{u}\cdot\nabla f(a) = \underbrace{|\mathbf{u}|}_{=1}\,|\nabla f(a)|\cos(\theta)$ är skalära projektionen av $\nabla f(a)$ på $\mathbf{u}$.



$$D_u f(a) = |\nabla f(a)| \cos(\theta)$$

<u>maximal</u> då $\cos(\theta)=1$, dvs
då u parallell med $\nabla f(a)$

<u>minimal</u> då $\cos(\theta)=-1$, dvs
då u antiparallell med $\nabla f(a)$

Alltså: största derivatan
fås i riktningen $u = \dfrac{\nabla f(a)}{|\nabla f(a)|}$

och den är $D_u f(a) = |\nabla f(a)|$.