MATHEMATICAL COMPETENCIES AND THE LEARNING OF MATHEMATICS: THE DANISH KOM PROJECT

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ABSTRACT

This paper presents the Danish KOM project (KOM: Competencies and the Learning of Mathematics), initiated by the Ministry of Education and other official bodies in order to create a platform for in-depth reform of Danish mathematics education, from school to university. The author of the paper was appointed as the director of the project. Its final report was published in October 2002.

The fundamental idea of the project is to base the description of mathematics curricula primarily on the notion of a "mathematical competency", rather than on syllabi in the traditional sense of lists of topics, concepts, and results. This allows for an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level.

"One should ask whose knowledge is best, not who knows the most"
(Montaigne, "On pedagogy", in Essays, 1st Book, Chapter 25 [26])

INTRODUCTION

In Danish mathematics education we have a number of *problems* and *challenges* at all educational levels, from school to university. Let me point out just a few of them.

Some are related to what we may call *the justification problem,* which manifests itself both at the level of society and at the level of the individual, At the former level, society needs a well educated population, to actively contribute to the shaping of society, and a broadly qualified work force, all of whom are able to activate mathematical knowledge, insights, and skills in a variety of situations and contexts. Yet, to an increasing extent young people opt away from educational programmes with a strong component of mathematics. At the individual level this is reflected in the socalled "relevance paradox": Even though mathematical knowledge is highly relevant in and to society, many, if not most, people have increasing difficulty at seeing that mathematics is relevant to them, as individuals.

At the same time there are counter currents in more and more western countries, Denmark included. It may well be said that the predominant international trend in mathematics education since World War II can be characterised as "mathematics for all". For a number of reasons this trend is now being challenged from different quarters. Firstly because it has to be admitted that by and large mathematics educators in western societies have not been too successful at really equipping the majority of the population with the mathematical knowledge, insights, and skills that are asked for. Bluntly put, many claim that the development towards mathematics for all has proved a bit of a failure, at least partly. This has led some mathematicians and mathematics educators to question the overall plot and to suggest that we should reserve serious mathematics education for the relatively few who can benefit from it at a reasonable investment of time and effort, while lowering the level of ambition with respect to the majority, in the hope that in this way we can avoid diluting the mathematics education of the former (selling away the crown jewels of mathematics) and avoid bringing excessive pain to the latter. This is in accordance with what can be found amongst an increasing number of industrialist, politicians and participants in public debates who fundamentally question the utility of mathematics to the general citizen in an era of computers, calculators and other technology. In other words, a threat to the "mathematics for all" movement is gaining momentum. Whatever position one wants to take to this threat it does require close attention.

Another category of problems and challenges are to do with what actually happens once it has been settled who in society is going to receive mathematics education. We call these problems and challenges *implementation problems*, for short. In Denmark there are several of these. I shall confine myself to indicating some of the main ones.

Mathematics teachers in Denmark form a very varied and heterogenous group. Teachers of the primary and lower secondary levels (grades K-9) are trained in teacher training colleges without any affiliation to the university system. They are trained as pedagogical generalists with some subject matter specialisation. However, only a minority specialise in mathematics, and besides, this specialisation is not very deep. This implies that a fair amount of the mathematics teaching delivered at levels K-9 is given by teachers who have a general orientation without much of a background in mathematics or the didactics of mathematics, but do have a broad background in general education and pedagogy. Sometimes it is stated - somewhat exaggeratedly, perhaps - that teachers at these levels are ambassadors of the student to the subjects. In contrast, mathematics teachers for the upper secondary levels (grades 9-12) and tertiary levels, including universities, mostly have a rather solid background in academic mathematics. These teachers have to have a least a university master's degree representing 5-7 years of study in mathematics and in one other subject. Many of them did not enter university studies of mathematics with the aim of becoming teachers. Rather they saw themselves as prospective researchers, or as working in industry (in a broad sense) on application problems etc. So, many upper secondary and tertiary teachers of mathematics have a rather poor background in the didactics and pedagogy of mathematics. At the university level one can even find teachers who simply discard the relevance of such a background to their profession. Pointedly stated, many of the university graduates who end up teaching mathematics see themselves as ambassadors of mathematics to the student. (It ought to be mentioned, though, that this depicts the overall state of affairs in Denmark at the moment. But things are changing in various ways, and in 5-10 years this picture is no longer likely to provide a fair representation of the situation.)

In summary, a large number *mathematics teachers* do not have an optimal background on which to exercise their profession, because of an insufficient preparation in mathematics or in the didactics and pedagogy of mathematics. This is not only a problem in itself, the variation mentioned in the backgrounds of various groups of mathematics teachers also result in cultural differences between these groups. In fact, such differences are special instances of larger cultural differences that exist between the various segments of the education system in Denmark. One could even speak of (semi-)closed circuits, each with their own culture and characteristics, one covering the primary and lower secondary levels and the teacher training colleges, one covering the upper secondary youth levels, grades 9-12, and one covering the tertiary, including university, levels. Moreover, each level operates within its own institutional framework. The differences mentioned give rise to severe *transition problems* between the various levels of the education system, above all from the lower to the upper secondary level, and from the upper secondary level to the tertiary level, in particular the universities. Students move, in mathematics, from one type of institution with its characteristic culture to another type with another culture, which produces marked discontinuities in the transition process.

A further aspect of the cultural and institutional differences that exist in Danish mathematics education is that mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system. To students, the aims and characteristics of the subject, and the rules of the game, appear to change markedly with the level. For instance, the roles of applications and modelling and of proof and proving, respectively, vary considerably across educational levels. In other words, there are problems with *the identity and coherence of mathematics* as a subject across the levels. The main problem is that the different educational levels tend to see themselves as competitors rather than as agents - acting at different sections of the education system - of the same overall endeavour and a common project, namely to increase and strengthen the mathematical competence of all students who receive some form of mathematics education.

Against this background, it is no wonder that it is difficult to pursue, identify, characterise and measure *progression* in students' mastering of mathematics.

What do we mean by progression if we do not agree on what we mean by mathematics and its mastery? This is closely related to another, but somewhat wider, problem, the *assessment problem*. The assessment problem consists of two parts. Firstly, there is the issue of interpretation, i.e. the problem to validly and reliably assess what we perceive as the key components of mathematical mastery. This is a matter of designing and adopting assessment instruments that are capable of telling us what we really want to know about students' knowledge, insights, and skills in, with and about mathematics. And the use of these instruments should not give rise to misleading results when we draw conclusions about students' mathematical competence. Here the problem is that quite a few of the assessment modes and instruments in actual use throughout the world, and in Denmark, do in fact produce misleading results, mostly because of insufficient validity which is often sacrificed for the benefit of reliability. Secondly, there is the problem of a frequent mismatch between the assessment modes we employ and the prevalent goals and forms of teaching and learning in the mathematics classrooms of our time.

Well, we have other problems and challenges in mathematics education in Denmark that I will have to leave untouched here (e.g. the ones generated by a considerable student heterogeneity in our classrooms at all levels). Suffice the problems and challenges mentioned to suggest that they are serious enough to deserve our full attention: "That is the question: whether 'tis nobler in the mind to suffer the slings and arrows of outrageous fortune, *or to take arms against the sea of troubles, and by opposing end them?*[...]" (Shakespeare: Hamlet, Act 3, Scene I).

THE DANISH KOM PROJECT

In an attempt to deal with problems and challenges such as the ones outlined above, a committee was appointed in Denmark in 2000, by the Ministry of Education and other official bodies, to conduct a project to explore the terrain of mathematics teaching and learning and to see what could be done to improve the state of affairs. The project was given the name 'the KOM project' (KOM – in Danish - stands for "Competencies and the Learning of Mathematics"). The Committee, which was chaired by the author of this paper, published its official report in October 2002 (Niss & Jensen, 2002). The terms of reference for the project were formulated by means of a series of questions as follows:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?

• What should be the content of up-to-date mathematics curricula?
• How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
• What does society demand and expect of mathematics teaching and learning?
• What will mathematical teaching materials look like in the future?
• How can we, in Denmark, make use of international experiences with mathematics teaching?
• How should mathematics teaching be organised in the future?

One, amongst several, intentions with the project was that it should act as a spearhead project for reform of the major subjects in the Danish education system. Since the initiation of the project, similar projects in Danish, the Sciences, and Foreign Languages have been initiated.

The Committee had twelve members, mathematicians, mathematics teachers, researchers in mathematics education, and a few people from outside of mathematics. The Committee soon decided to appoint a group of twenty-odd "sparring partners", representing all segments of mathematics education in Denmark, whose task it was to comment on the work of the Committee along the way. Moreover, the Chair, the Secretary, and other members of the Committee have presented and discussed the project at several dozens of meetings with mathematics teachers and others around the country. The idea was to ensure as much as possible of co-ownership for ordinary teachers, schools, organisations and institutions, thus trying to avoid the well known trap of being yet another top-down reform project that fails exactly because those who are meant to implement it do not feel ownership to it. There are thousands of subtle ways, at least in Denmark, to undermine reforms which the key agents are against, without formally attacking it or breaking the rules set from above.

The Committee based its work on an attempt to answer the following question:

What does it mean to master mathematics?

If that question could be answered properly we would possess a means by which the other issues of the project could subsequently be addressed.

To illustrate the endeavour, let me offer an analogy. What do we mean by 'literacy'[1)], i.e. to master a language and use it in context? I submit that to master a language consists in being able to

• understand and interpret other people's oral speech
• understand and interpret written texts produced by others

and to

- speak and express oneself orally
- express oneself in writing,

and all of this in a variety of different linguistic registers, and with reference to a variety of different forms and domains of oral and written “texts”. It is essential to keep in mind that the main constituents of literacy are the same for first graders and professors of literature, but the constituents manifest themselves quite differently in a 1st grade and in the university.

It should be noted that mastering a language certainly requires, but definitely cannot be reduced to, factual knowledge and skills concerning orthography, vocabulary, grammar etc.

Now what is the counterpart in mathematics of mastering a language? That is the question we shall address in this paper as we did in the KOM project. Before going into details with this question it should be made clear that the KOM project was not designed to be a research project in the traditional sense. It already follows from the terms of reference that no single project could ever attempt to answer, in a scientifically sound way, all these broad and difficult questions, each of which may deserve a research project of its own. Rather, the KOM project may be described as an *analytical development project*. Its task was to produce thoughtful (we hope!) analyses of the *problématique* outlined by the terms of reference, to make recommendations for reform in mathematics education in Denmark, and to provide ideas and inspiration for the further development of mathematics teaching and learning in Denmark.

Experiences suggest that it is worth also stating what the project is *not* supposed to be. It is not supposed to justify the presence of mathematics in education system for various groups of recipients, i.e. to answer the question ‘why mathematics education?’, even if this question is terribly important in its own right. Also it is not supposed to be a legislative project in the direct sense of proposing ready-made curricula to be installed at all educational levels or very specific structural reforms. It is, though, certainly the intention that the project should lead to legislative action at various levels, but it is left to the respective authorities themselves to take such action. Finally, the project is not meant to be an implementation project to design and orchestrate how mathematics education should be organised and conducted in the different segments of the education system. That, too, has to be left to those in charge of such implementation at their platforms of operation.

MATHEMATICAL COMPETENCE AND COMPETENCIES

To master mathematics means to posses mathematical competence. But then, what is that?

To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. *Mathematical competence* then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills, in the same way as vocabulary, orthography, and grammar are necessary but not sufficient prerequisites for literacy.

A mathematical competency is a clearly recognisable and distinct, major constituent of mathematical competence.

In the project we have adopted an attempt made by the author of this paper (Niss, 1999) to identify these competencies. There are eight competencies which can be said to form two groups. The first group of competencies are to do with the ability to *ask and answer questions in and with mathematics*:

1. Thinking mathematically (mastering mathematical modes of thought) such as

- *posing questions* that are characteristic of mathematics, and *knowing the kinds* of answers (not necessarily the answers themselves or how to obtain them) that mathematics may offer;
- understanding and handling the *scope* and *limitations* of a given *concept*.
- *extending* the scope of a *concept* by *abstracting* some of its properties; *generalising results* to larger classes of objects;
- *distinguishing* between different *kinds of mathematical statements* (including conditioned assertions ('if-then'), quantifier laden statements, assumptions, definitions, theorems, conjectures, cases):

2. Posing and solving mathematical problems
such as

- *identifying*, *posing*, and *specifying* different kinds of mathematical *problems* – pure or applied; open-ended or closed;
- *solving* different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways.

3. Modelling mathematically (i.e. analysing and building models)
such as

- *analysing* foundations and properties of *existing models*, including assessing their range and validity
- *decoding* existing models, i.e. translating and interpreting model elements in terms of the 'reality' modelled
- *performing active modelling* in a given context
  - structuring the field
  - mathematising
  - working with(in) the model, including solving the problems it gives rise to

- validating the model, internally and externally
- analysing and criticising the model, in itself and vis-à-vis possible alternatives
- communicating about the model and its results
- monitoring and controlling the entire modelling process.

4. Reasoning mathematically
such as
• f*ollowing* and *assessing chains of arguments*, put forward by others
• *knowing* what a mathematical *proof is* (not), ands how it differs from other kinds of mathematical reasoning, e.g. heuristics
• *uncovering* the *basic ideas* in a given line of argument (especially a proof), including distinguishing main lines from details, ideas from technicalities;
• *devising* formal and informal mathematical *arguments*, and *transforming* heuristic arguments to valid proofs, i.e. *proving statements.*

The other group of competencies are to do with the ability to deal with and *manage mathematical language and tools:*

5. Representing mathematical entities (objects and situations)
such as
• *understanding* and *utilising* (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations;
• understanding and utilising the *relations between different representations* of the same entity, including knowing about their relative strengths and limitations;
• *choosing* and *switching* between representations.

6. Handling mathematical symbols and formalisms
such as
• *decoding* and *interpreting symbolic and formal* mathematical *language,* and understanding *its relations to natural language*;
• understanding the *nature* and *rules* of *formal mathematical systems* (both syntax and semantics);
• *translating* from *natural language* to *formal/symbolic language*
• *handling* and manipulating statements and *expressions* containing *symbols* and *formulae*.

7. Communicating in, with, and about mathematics
such as
• *understanding others'* written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content;
• *expressing oneself*, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters.

8. Making use of aids and tools (IT included)
such as
• *knowing* the *existence* and *properties* of various tools and aids for mathematical activity, and their range and limitations;
• being able to *reflectively use* such aids and tools.

A number of comments are in order.

All these eight competencies are to do with mental or physical processes, activities, and behaviours. In other words, the focus is on what individuals can do. This makes the competencies behavioural (not to mistake for behavioristic).

The competencies are closely related - they form a continuum of overlapping clusters - yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint.

All competencies have a dual nature, as they have an analytical and a productive aspect. The analytical aspect of a competency focuses on understanding, interpreting, examing, and assessing mathematical phenomena and processes, such as, for instance, following an controlling a chain of mathematical arguments or understanding the nature and use of some mathematical representation, whereas the productive aspect focuses on the active construction or carrying out of processes, such as inventing a chain of arguments or activating and employing some mathematical representation in a given situation.

Furthermore, although the competencies are formulated in terms that may apply to other subjects as well, these terms are here to be understood in a strict mathematical sense. Thus we are talking about *mathematical* representations, not representations in general. Similarly, we are talking about *mathematical* reasoning, including proof and proving, not about reasoning in general like in general logic or in a court room, and we are talking about *mathematical* symbols, not other kinds of symbols such as icons or chemical symbols, let alone religious or literary symbols. In other words the competencies are specific to mathematics.

Yet they are overarching across mathematical topic areas and educational levels, i.e. they are not tied to specific topics, curricula or classrooms. However, they do indeed manifest themselves quite differently at different levels and in different countries, exactly as is the case with mastering a language.

When examing how well the competencies cover our common, albeit somewhat vague, notions of mathematical competence at large, one can often hear people ask where, say, mathematical intuition, creativity and the ability to deal with abstraction come in. Why are they not listed as independent competencies on a par with the others? A detailed answer to this question will carry us too far, but a more general answer is that they are subsumed under some or all of the eight competencies. For example, creativity can be perceived as the union of all the productive aspects of the competencies. Similarly, the ability to deal with abstraction forms part of all competencies, as does mathematical intuition.

All this is not to say that the competencies are supposed to constitute a canonical system to which there are no alternatives. Of course, mathematical competence could probably be conceptualised by a different set of components. It just so happens that the present set seems to be able to capture the essential aspects of mathematical mastery reasonably well.

A particularly important comment is to do with the relationship between the competencies and mathematical subject matter. A mathematical competency can only be developed and exercised in dealing with such subject matter. Yet, the choice of curriculum topics does not follow from the focus on the competencies. Rather the competencies and mathematical topic areas are to be seen as orthogonal. This implies that the relationship can suitably by represented by a matrix whose rows are the topics chosen for the educational level at issue and, and whose columns are the eight competencies. Then each cell specifies how the corresponding competency manifests itself when dealing with the corresponding topic at the educational level at issue.

MASTERY OF COMPETENCIES: ASSESSMENT AND PROGRESSION

Possessing a mathematical competency (to some degree) consists in being prepared and able to act mathematically on the basis of knowledge and insight. The actions at issue can be both physical, behavioural (including linguistic) and mental. So, a valid evaluation of an individual's mathematical competencies has to be founded on the identification of the presence and range of his or her competencies in relation to mathematical activities in which the individual is or has been involved. The carrying through of any mathematical activity requires the exercise of one or several mathematical competencies. Therefore it becomes an essential task to identify – a priori as well as a posteriori – necessary competencies and sufficient competencies involved in a variety of mathematical activities such as solving a pure or applied mathematical problem, reading a mathematical text, proving a theorem, investigating the structure of a mathematical theory, writing a text containing mathematical components, giving a talk etc.

An individual's possession of a given mathematical competency has three dimensions:

The *degree of coverage* is the extent to which the person masters the characteristic aspects of the competence at issue as indicated in the above characterisation of it.

The *radius of action* indicates the spectrum of contexts and situations in which the person can activate that competence.

The *technical level* indicates how conceptually and technically advanced the entities and tools are with which the person can activate the competence.

Each dimension represents a non-quantitative, partial ordering. Nevertheless, in metaphorical terms we can think of the individual's possession of the competency as a three-dimensional box. The (metaphorical) volume of the competency is the "product" of the degree of coverage, the radius of action, and the technical level. This suffices to suggest that if one of the dimensions has measure zero, the same is true with the volume of the competency. It also suggests that the "same" volume can be obtained by infinitely many different combinations of the three measures.

Suppose that we are able to gauge each dimension of someone's mastery of a given competency at a given point in time. Then we would also be able to trace the development of those dimensions over time, which is just another way of identifying and monitoring progression. Progression of an individual's possession of a competency is simply growth with respect to one or more of these dimensions.

This leaves us with the fundamental question of how to gauge someone's mastery of a mathematical competency, which is the key issue in the assessment of competencies. Due to the limitations of this paper we will have to leave this highly significant issue here. Let us confine ourselves to mentioning that no single assessment form and instrument is sufficient to validly and reliably assess the entire spectrum of mathematical competencies. Moreover, often a given activity gives rise to only some of the competencies, and different activities will involve different sets of competencies. So, in order for assessment to provide a fair and comprehensive coverage of the entire set of mathematical competencies, a board spectrum of activities are needed.

OVERVIEW AND JUDGEMENT REGARDING MATHEMATICS AS A DISCIPLINE

A mathematical competency is activated in situations which contain actual or potential mathematical challenges. In addition to the eight competencies, we have found it essential to also focus on mathematics as a discipline. More specifically, we have identified three kinds of overview and judgement regarding mathematics as a discipline that students should develop throughout their study of mathematics. These are overview and judgement concerning

- the *actual application* of mathematics in other subjects and fields of practice, of scientific or societal significance;
- the *historical development* of mathematics, internally as well as externally;
- the special *nature of mathematics* as a discipline.

Needless to say, these kinds of overview and judgement are closely related to the possession of the mathematical competencies, but they cannot be derived from them. As is the case with the competencies, the three kinds of overview and judgement are comprehensive, overarching, i.e. not tied to specific mathematical content or to specific educational levels. In other words they are general to mathematics as well as being specific to mathematics.

CONCLUSION

The competencies and the three kinds of overview and judgement can be used in different ways in mathematics education.

Firstly, they can be employed for *normative* purposes, e.g. with respect to specification of a curriculum or of desired outcomes of student learning. In other words, they provide a tool for clarifying, in a non-circular way, how we want mathematical education to function.

Secondly, they can be used for *descriptive* purposes. More specifically, they can be used to describe and characterise actual teaching practice, what happens in classrooms, what is being pursued in testing and examinations, and the actual outcomes of students' learning. They can also be used to compare different mathematics curricula and different kinds of mathematics education at different levels or in different places, and so forth.

Finally, by being explicit instruments of characterisation they can also be used as *meta-cognitive support* for teachers and students by assisting them to clarify, monitor and control their teaching and learning, respectively.

Many aspects of the KOM project have had to be left untouched in this paper, above all the essential issue of teacher education. How can we educate teachers, for all educational levels, who can foster the development of the eight competencies and the three kinds of overview and judgement with students? In the project we have attempted at characterising the competencies of "the excellent mathematics teacher". But this is another chapter of our study which has to await another occasion to be told.

1) It should be noted that the thinking behind and before the Danish KOM-project has influenced the mathematics domain of OECD's PISA project, partly because the author is a member of the mathematics expert group for that project. That influence is reflected in PISA's notion of mathematical literacy and its constituents. (OECD, 1999)

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