Problem 5.1

(a) $$\frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x - 2\cdot\sin x\cos x}{x^3}$$

$$= \frac{2\cdot \sin x \cdot (1-\cos x)}{x \cdot x^2} \to 2\cdot 1\cdot\frac{1}{2} = 1 \text{ då } x\to 0$$

$\frac{\sin x}{x} \to 1$ då $x\to 0$, $\boxed{\frac{1-\cos x}{x^2}} \to \frac{1}{2}$ då $x\to 0$

(b) $$\frac{3-4\cos x+\cos 2x}{x^4} = \frac{3-4\cos x+2\cos^2 x - 1}{x^4}$$

$$= \frac{2\cos^2 x - 4\cos x + 2}{x^4} = \frac{2\cdot(\cos^2 x - 2\cos x + 1)}{x^4}$$

$$= \left| \text{KVADRERINGS-REGELN} \right| = \frac{2\cdot(\cos x - 1)^2}{x^4} = 2\cdot\left(\frac{1-\cos x}{x^2}\right)^2$$

$\frac{1-\cos x}{x^2} \to \frac{1}{2}$ då $x\to 0$

$$\to 2\cdot\left(\frac{1}{2}\right)^2 = \frac{2}{4} = \frac{1}{2} \text{ då } x\to 0$$

(c) $$\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\cdot\tan x = \left| \begin{matrix}\text{subst.}\\ z = x - \frac{\pi}{2}\\ z\to 0 \Leftrightarrow x\to \pi/2\end{matrix} \right| =$$

$$= \lim_{z\to 0} z\cdot\tan\left(z+\frac{\pi}{2}\right) = \lim_{z\to 0} z\cdot\frac{\sin(z+\pi/2)}{\cos(z+\pi/2)} = \left| \text{ADDITIONS-FORMLERNA} \right|$$

$$= \lim_{z\to 0} z\cdot\frac{\cos z}{-\sin z} = \lim_{z\to 0}\frac{-\cos z}{\frac{\sin z}{z}} = \frac{-\cos 0}{1} = \frac{-1}{1} = -1$$

$\frac{\sin z}{z} \to 1$ då $z\to 0$

Problem 5.2

(a) $$x^m - 1 = x^m - 1^m = \left| \begin{matrix}\text{generaliserade}\\ \text{konjugatregeln}\end{matrix} \right| =$$

$$= (x-1)\cdot\underbrace{\left(x^{m-1}\cdot 1^0 + x^{m-2}\cdot 1^1 + x^{m-3}\cdot 1^2 + \dots + x^2\cdot 1^{m-3} + x^1\cdot 1^{m-2} + x^0\cdot 1^{m-1}\right)}_{m \text{ stycken termer}}$$

$$= (x-1)\left(x^{m-1} + x^{m-2} + \dots + x + 1\right)$$

$$\lim_{x\to 1} \frac{x^m-1}{x^n-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}\cdot \overbrace{(x^{m-1}+x^{m-2}+\dots+x^1+1)}^{m \text{ stycken termer}}}{\cancel{(x-1)}\underbrace{(x^{n-1}+x^{n-2}+\dots+x^1+1)}_{n \text{ stycken termer}}} =$$

$$= \frac{1^{m-1}+1^{m-2}+\dots+1^1+1}{1^{n-1}+1^{n-2}+\dots+1^1+1} = \frac{m\cdot 1}{n\cdot 1} = \underline{\underline{\frac{m}{n}}}$$

(b) $$\lim_{x\to 1} \frac{x^{m/p}-1}{x^{n/q}-1} = \left/ \text{variabelbyte } \begin{matrix} x = t^{pq} \\ t = x^{1/pq} \end{matrix} \right/$$

$$x\to 1 \overset{x,t>0}{\Longleftrightarrow} t \to 1$$

$$= \lim_{t\to 1} \frac{(t^{pq})^{m/p}-1}{(t^{pq})^{n/q}-1} = \lim_{t\to 1} \frac{t^{mq}-1}{t^{np}-1} = /(a)/ = \underline{\underline{\frac{mq}{np}}}$$

(c) $\alpha, \beta \in \mathbb{R}$, nollskilda. $\lim_{x\to 1} \frac{x^\alpha-1}{x^\beta-1} = ?$

basbyte för exponentialfunktionen ger

$$x^\alpha = (e^{\ln x})^\alpha = e^{\alpha\cdot \ln x}$$

$$\therefore \quad \frac{x^\alpha-1}{x^\beta-1} = \frac{e^{\alpha\cdot\ln x}-1}{e^{\beta\cdot \ln x}-1} = \frac{e^{\alpha\cdot\ln x}-1}{\alpha \ln x}\cdot(\alpha \ln x)\cdot$$

$$\cdot \frac{\beta \ln x}{e^{\beta\ln x}-1}\cdot\frac{1}{\beta\ln x}$$

$$= \frac{\alpha}{\beta}\cdot\frac{e^{\alpha\ln x}-1}{\alpha\ln x}\cdot\frac{\beta\ln x}{e^{\beta\ln x}-1}$$

$$\lim_{x\to 1}\frac{x^\alpha-1}{x^\beta-1} = \frac{\alpha}{\beta}\cdot\lim_{x\to 1}\frac{e^{\alpha\ln x}-1}{\alpha\ln x}\cdot\lim_{x\to 1}\frac{\beta\ln x}{e^{\beta\ln x}-1}$$

$$= \left/ \text{variabelbyte } \begin{matrix} y=\alpha\ln x, & x\to 1 \Leftrightarrow y\to 0 \\ z=\beta\ln x, & x\to 1\Leftrightarrow z\to 0\end{matrix}\right/$$

$$= \frac{\alpha}{\beta}\cdot\lim_{y\to 0}\frac{e^y-1}{y}\cdot\lim_{z\to 0}\frac{z}{e^z-1} = \frac{\alpha}{\beta}\cdot 1\cdot 1$$

$$= \underline{\underline{\frac{\alpha}{\beta}}}$$

Problem 5.3

(a)

$$(n+4)^{27} = n^{27} + 27 \cdot n^{26} \cdot 4 + P_{25}(n)$$

$$(n+1)^{27} = n^{27} + 27 \cdot n^{26} \cdot 1 + \tilde{P}_{25}(n)$$

polynom med gradtal = 25

$$\therefore \quad (n+4)^{27} - (n+1)^{27} = 27 \cdot n^{26} \cdot 4 - 27 \cdot n^{6} \cdot 1 + P(n)$$

$$= 81\, n^{26} + P(n)$$

$$(2n^2+5)^{13} = (2n^2)^{13} + Q_{24}(n) = 2^{13} n^{26} + Q_{24}(n)$$

$$= 8192\, n^{26} + Q_{24}(n)$$

$$(n^2-1)^{13} = (n^2)^{13} + \tilde{Q}_{24}(n) = n^{26} + \tilde{Q}_{24}(n)$$

polynom med gradtal = 24

$$\therefore \quad (2n^2+5)^{13} - (n^2-1)^{13} = 8192\, n^{26} - n^{26} + Q(n)$$

$$= 8191\, n^{26} + Q(n)$$

$$\lim_{n\to\infty} \frac{(n+4)^{27} - (n+1)^{27}}{(2n^2+5)^{13} - (n^2-1)^{13}} = \lim_{n\to\infty} \frac{81\, n^{26} + P(n)}{8191\, n^{26} + Q(n)}$$

$$= \lim_{n\to\infty} \frac{81 + \frac{P(n)}{n^{26}}}{8191 + \frac{Q(n)}{n^{26}}} = \frac{81+0}{8191+0} = \underline{\underline{\frac{81}{8191}}}$$

$\frac{P(n)}{n^{26}} \to 0$, $\frac{Q(n)}{n^{26}} \to 0$

(b)

$$\lim_{n\to\infty} \frac{(n^3 \cdot \cos n + n^2 \sin n)^{100} - (n^3 \cdot \sin n + n^2)^{100}}{(n^9 + n^5 \cos n + n^2 \sin n)^{34}}$$

$$= \lim_{n\to\infty} \frac{n^{3\cdot 100}\left(\left(\cos n + \frac{\sin n}{n}\right) - \left(\sin n + \frac{1}{n}\right)\right)^{100}}{n^{9\cdot 34}\left(1 + \frac{\cos n}{n^4} + \frac{\sin n}{n^7}\right)^{34}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{n^6}}_{\to 0} \cdot \underbrace{\frac{\left(\left(\cos n + \frac{\sin n}{n}\right) - \left(\sin n + \frac{1}{n}\right)\right)^{100}}{\left(1 + \frac{\cos n}{n^4} + \frac{\sin n}{n^7}\right)^{34}}}_{\text{instängd mellan } 0 \text{ o } \frac{4^{100}}{0{,}5^{34}}}$$

(väldigt grov uppskattning)

$= \underline{0}$ enligt instängningsregeln

Problem 5.4

Basbyte för exponentialfunktionen ger att

$$7^{\ln x} = \left(e^{\ln 7}\right)^{\ln x} = e^{\ln 7 \cdot \ln x} = \left(e^{\ln x}\right)^{\ln 7} = x^{\ln 7}$$

$$\therefore \quad \frac{7^{\ln x}}{x^a} - bx = x^{\ln 7 - a} - bx =: f(x)$$

Fall 1: om $a = \ln 7$:

$$f(x) = x^0 - bx = 1 - bx \quad \begin{cases} b > 0 \Rightarrow f(x) \to -\infty \\ b = 0 \Rightarrow f(x) \to 1 \\ b < 0 \Rightarrow f(x) \to +\infty \end{cases}$$

då $x \to \infty$

Fall 2: om $a \neq \ln 7$:

$$f(x) = x^{\ln 7 - a} - bx = x \cdot \left(x^{\ln 7 - a - 1} - b\right)$$

(i) om $a > \ln 7 - 1 = \ln\left(\frac{7}{e}\right)$, så är det "$bx$" som är dominerande:

$$\begin{cases} b > 0 & f(x) \to -\infty \\ b = 0 & f(x) \to 0 \\ b < 0 & f(x) \to +\infty \end{cases} \qquad \text{då } x \to \infty$$

(ii) om $a = \ln 7 - 1 = \ln\left(\frac{7}{e}\right)$, så $f(x) = x - bx = x \cdot (1-b)$

$$\begin{cases} b > 1 \Rightarrow f(x) \to -\infty \\ b = 1 \Rightarrow f(x) = 0 \\ b < 1 \Rightarrow f(x) \to +\infty \end{cases} \qquad \text{då } x \to \infty$$

(iii) om $a < \ln 7 - 1 = \ln\left(\frac{7}{e}\right)$, så är det "$x^{\ln 7 - a}$" som är dominerande

$$\Rightarrow f(x) \to \infty \quad \text{då } x \to \infty$$

(oavsett b)