Lösningar tentamen MVE425c 18/3 - 17

1. (a) $D(x^2 \cdot \arctan(x^2)) =$

$$= 2x \cdot \arctan(x^2) + x^2 \frac{1}{1+(x^2)^2} \cdot D(x^2) =$$

$$= 2x \cdot \arctan(x^2) + \frac{2x^3}{1+x^4}$$

(b) $D\left(e^{-x^2-x}(2x+1)^2\right) =$

$$= e^{-x^2-x} \cdot D(-x^2-x) \cdot (2x+1)^2 + e^{-x^2-x} 2(2x+1) \cdot D(2x+1)$$

$$= e^{-x^2-x}(2x+1)\left((-2x-1)(2x+1)+4\right) =$$

$$= e^{-x^2-x}(2x+1)\left(4-(2x+1)^2\right)$$

(c) $D\left(\frac{1}{\ln(x^2+1)}\right) = D\left(\left(\ln(x^2+1)\right)^{-1}\right) =$

$$= -1 \cdot \left(\ln(x^2+1)\right)^{-2} \cdot D\ln(x^2+1) =$$

$$= -\frac{1}{\left(\ln(x^2+1)\right)^2} \cdot \frac{1}{x^2+1} \cdot D(x^2+1) =$$

$$= -\frac{2x}{(x^2+1)\left(\ln(x^2+1)\right)^2}$$

2. Tänk $y = y(x)$ och derivera båda leden m.a.p. x:

$$\underbrace{2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y'}_{= D(x^2 y^3)} = 8y^3 \cdot y' + 3x^2$$

I punkten $(-1, 1)$:

$$2 \cdot (-1) \cdot 1^3 + (-1)^2 \cdot 3 \cdot 1^2 \cdot y' = 8 \cdot 1^3 \cdot y' + 3 \cdot (-1)^2$$

$$\Leftrightarrow -2 + 3y' = 8y' + 3 \Leftrightarrow 5y' = -5$$

$$\Rightarrow k_T = y'\Big|_{(-1,1)} = -1$$

Vet att: $k_T \cdot k_N = -1 \Leftrightarrow k_N = \dfrac{-1}{k_T} = 1$

$$\therefore \; N: \; y - 1 = 1 \cdot (x - (-1)) \Leftrightarrow$$

$$\Leftrightarrow y = x + 2$$

3. Steg 1: $D_f = \mathbb{R} \setminus \{\pm\sqrt{2}\}$

Steg 2: I. Lodräta asymptoter:

$f(x) \to \infty$ då $x \to \sqrt{2}^+$ och $x \to -\sqrt{2}^-$

$f(x) \to -\infty$ då $x \to \sqrt{2}^-$ och $x \to -\sqrt{2}^+$

$\Rightarrow$ $x = -\sqrt{2}$ och $x = \sqrt{2}$ lodräta asymptoter

II. Vågräta asymptoter:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^{2x}}{x^2 - 2} = \infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \underbrace{e^{2x}}_{\to 0} \cdot \frac{1}{\underbrace{x^2 - 2}_{\to 0}} = 0$$

$\Rightarrow$ $y = 0$ ~~lodräta~~ vågrät asymptot

III. Sneda asymptoter:

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{e^{2x}}{x(x^2 - 2)} = \infty$$

$\Rightarrow$ Inga sneda asymptoter

Steg 3:

$$f'(x) = \frac{2e^{2x} \cdot (x^2 - 2) - e^{2x} \cdot 2x}{(x^2 - 2)^2} =$$

$$= \frac{2e^{2x}(x^2-2-x)}{(x^2-2)^2} = \frac{2e^{2x}(x^2-x-2)}{(x^2-2)^2}$$

$$f'(x) = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow$$

$$\Rightarrow x = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = \frac{1}{2} \pm \frac{3}{2}$$

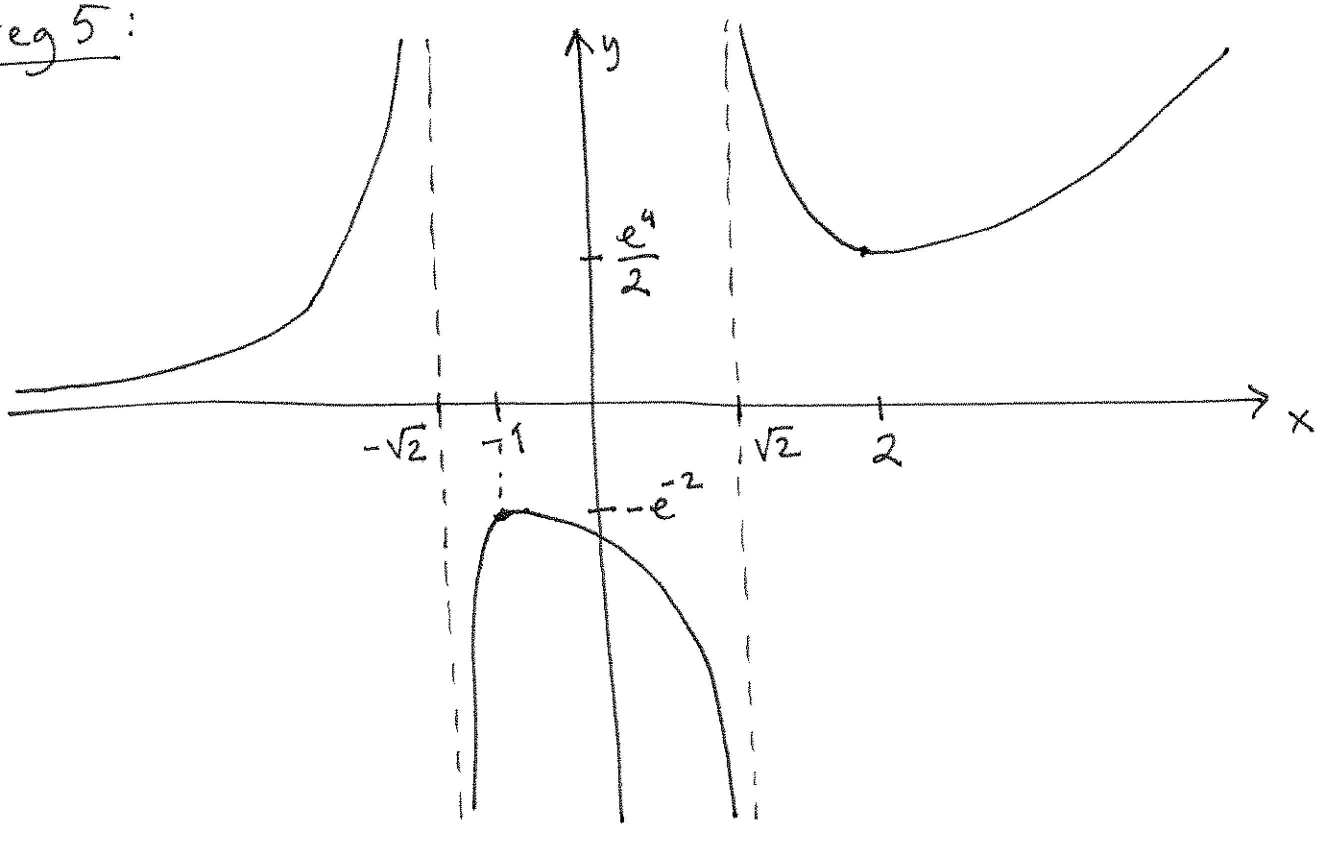
$$\Rightarrow x_1 = 2, \quad x_1 = -1 \quad \leftarrow \text{Kritiska punkter}$$

$$\therefore \quad f'(x) = \frac{2e^{2x}(x-2)(x+1)}{(x^2-2)^2}$$

Steg 4:

	$-\infty$		$-\sqrt{2}^-$	$-\sqrt{2}^+$		-1		$\sqrt{2}^-$	$\sqrt{2}^+$		2		∞
f'		$+$			$+$	0	$-$			$-$	0	$+$	
f	0	$\nearrow$	∞	$-\infty$	$\nearrow$	$-e^{-2}$	$\searrow$	$-\infty$	∞	$\searrow$	$\frac{e^4}{2}$	$\nearrow$	∞

Steg 5:



4. $D_f = (0,\infty)\setminus\{1\} = (0,1)\cup(1,\infty)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{\ln(x)} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\ln(x)} = \infty$$

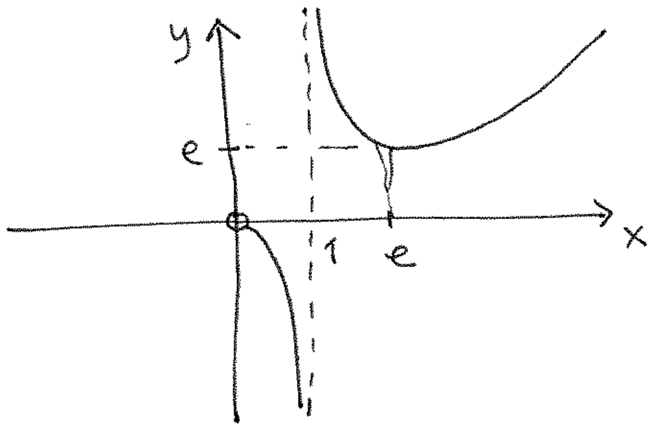
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{\ln(x)} = \infty \quad \left(\text{"}\frac{1}{0^+}\text{"}\right)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{\ln(x)} = -\infty \quad \left(\text{"}\frac{1}{0^-}\text{"}\right)$$

$$f'(x) = \frac{1\cdot\ln(x) - x\cdot\frac{1}{x}}{(\ln(x))^2} = \frac{\ln(x)-1}{(\ln(x))^2}$$

$$f'(x) = 0 \Rightarrow \ln(x) = 1 \Leftrightarrow x = e$$

	0^+		1^-	1^+		e		∞
f'		$-$			$-$	0	$+$	
f	0	$\searrow$	$-\infty$	∞	$\searrow$	e	$\nearrow$	∞



$\therefore\ V_f = (-\infty,0)\cup[e,\infty)$

5. $Ce^x = 1 + x \iff C = \dfrac{1+x}{e^x}$

$\therefore$ "Rita grafen till $f(x) = \dfrac{1+x}{e^x}$ "

$D_f = \mathbb{R}$

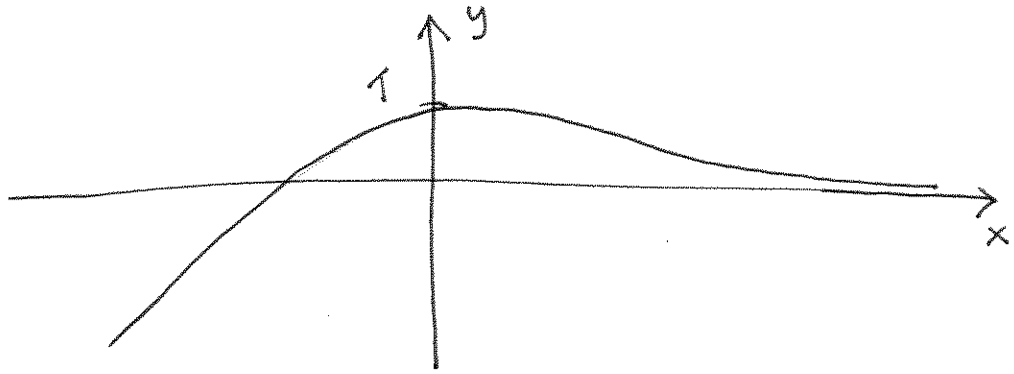
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+x}{e^x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \underbrace{\frac{1}{e^x}}_{\to\infty} \cdot \underbrace{(1+x)}_{\to-\infty} = -\infty$$

$$f'(x) = \frac{1\cdot e^x - (1+x)e^x}{(e^x)^2} = \frac{\cancel{e^x}(1-1-x)}{(e^x)^{2}} = -\frac{x}{e^x}$$

$f'(x) = 0 \Rightarrow x = 0 \leftarrow$ kritisk punkt

	$-\infty$		0		∞
f'		$+$		$-$	
f	$-\infty$	$\nearrow$	1	$\searrow$	0



$\therefore$ Inga lösningar då $C \in (1, \infty)$

2 lösningar då $C \in (0, 1)$

1 lösningar då $C \in (-\infty, 0] \cup \{1\}$

6. I en punkt x har tangenten till $y = \frac{1}{1+x^2}$ lutningen $y' = -\frac{2x}{(1+x^2)^2}$.

Låt $f(x) = -\frac{2x}{(1+x^2)^2}$. Vill maximera $f(x)$.

$D_f = \mathbb{R}$

$$f'(x) = -\frac{2(1+x^2)^2 - 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} =$$

$$= -\frac{2\cancel{(1+x^2)}(1+x^2-4x^2)}{(1+x^2)^{\cancel{4}\,3}} = \frac{2(3x^2-1)}{(1+x^2)^3}$$

$$f'(x) = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}$$

		$-1/\sqrt{3}$		$1/\sqrt{3}$	
f'	+	0	−	0	+
f	↗	lok.max.	↘	lok. min.	↗

$\therefore\ x_2 = -\frac{1}{\sqrt{3}}$ lokal max. punkt

$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{2\cdot(-1/\sqrt{3})}{\left(1+\frac{1}{3}\right)^2} = \frac{2}{\sqrt{3}\cdot\frac{4^2}{3^2}} = \frac{2\cdot 3\sqrt{3}}{16} = \frac{3\sqrt{3}}{8}$$

$\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ lokal eller global max?

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -\frac{2x}{(1+x^2)^2} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -\frac{2x}{(1+x^2)^2} = 0$$

$\therefore$ Tangenten till $y = \frac{1}{1+x^2}$ har störst positiv lutning då $x = -\frac{1}{\sqrt{3}}$. Den maximala lutningen är $\frac{3\sqrt{3}}{8}$.