MVE470/MVE351, 2017-08-25

1 (a) För $f(x,y) = 4x^2-16x+y^2+12$

har vi $\nabla f(x,y) = (8x-16,\ 2y)$

En normalvektor till nivåkurvan $f(x,y)=0$ i $(6/5, 6/5)$ är

$$\nabla f\left(\frac{6}{5}, \frac{6}{5}\right) = \left(\frac{48}{5}-16, \frac{12}{5}\right) = \left(-\frac{32}{5}, \frac{12}{5}\right)$$

Tangentlinjens ekvation är

$$-\frac{32}{5}\left(x-\frac{6}{5}\right) + \frac{12}{5}\left(y-\frac{6}{5}\right) = 0.$$

(b) $\frac{\partial f}{\partial x} = y - 3x^2y - y^3$ $\quad \frac{\partial f}{\partial y} = x - x^3 - 3xy^2$

$\frac{\partial^2 f}{\partial x \partial y} = 1 - 3x^2 - 3y^2$ $\quad \frac{\partial^2 f}{\partial x^2} = -6xy$

$\frac{\partial^2 f}{\partial y^2} = -6xy$

Taylorpolynomet $P(x,y) =$

$$= f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(1,1)(x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,1)(x-1)(y-1)\right.$$
$$\left.+ \frac{\partial^2 f}{\partial y^2}(1,1)(y-1)^2\right) =$$

$$= -1 - 3(x-1) - 3(y-1) +$$
$$+ \frac{1}{2}\left(-6(x-1)^2 - 10(x-1)(y-1) - 6(y-1)^2\right)$$

(c) $r'(t) = (-\sin t, \cos t, -2\sin 2t)$

$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4\sin^2 2t} = \sqrt{1+4\sin^2 2t}$

$|r'(t)|$ är störst då $\sin^2 2t = 1$

vilket ger $2t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\right\}$

dvs $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Partikelns acceleration =

$= r''(t) = (-\cos t, -\sin t, -4\cos 2t)$

$r''\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$, $r''\left(\frac{3\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$

$r''\left(\frac{5\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, $r''\left(\frac{7\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

(d) $\frac{d^2}{ds^2}(f(s^2, 2s+1)) =$

$= \frac{d}{ds}\left(\frac{d}{ds} f(s^2, 2s+1)\right) \overset{\text{kedjeregeln}}{=}$

$= \frac{d}{ds}\left(f_1'(s^2, 2s+1) \cdot 2s + f_2'(s^2, 2s+1) \cdot 2\right) \overset{\text{kedjeregeln + produktregeln}}{=}$

$= \left(f_{11}''(s^2, 2s+1) \cdot 2s + f_{12}''(s^2, 2s+1) \cdot 2\right) 2s +$

$+ f_1'(s^2, 2s+1) \cdot 2 + \left(f_{21}''(s^2, 2s+1) \cdot 2s + f_{22}''(s^2, 2s+1) \cdot 2\right) \cdot 2 =$

$= f_{11}''(s^2, 2s+1) \cdot 4s^2 + f_{12}''(s^2, 2s+1) \cdot 8s + f_{22}''(s^2, 2s+1) \cdot 4 +$

$+ f_1'(s^2, 2s+1) \cdot 2$

2.

(a) $$(Df)(x,y,z) = \begin{bmatrix} z & 0 & x \\ 0 & 2y\cos(yz) - y^2 z\sin(yz) & -y^3\sin(yz) \end{bmatrix}$$

$$Df(2,1,0) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

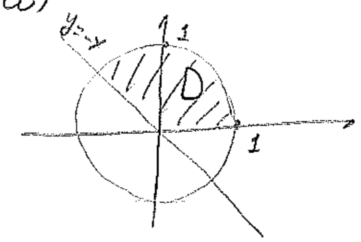
(b) $$f(1.9,\ 1.1,\ 0.2) \approx f(2,1,0) + Df(2,1,0)\begin{bmatrix} 1.9-2 \\ 1.1-1 \\ 0.2-0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} -0.1 \\ 0.1 \\ 0.2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 1.2 \end{bmatrix}$$

3.

(a)



Polar substitution

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad 0 \le r \le 1, \quad 0 \le \varphi \le \frac{3\pi}{4}$$

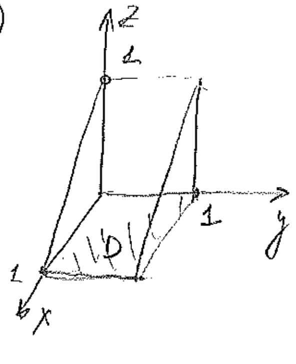
$$\iint_D \frac{1}{\sqrt{4-x^2-y^2}}\,dx\,dy = \int_0^1 \int_0^{3\pi/4} \frac{1}{\sqrt{4-r^2}}\, r\,d\varphi\,dr =$$

$$= \frac{3\pi}{4}\int_0^1 \frac{r}{\sqrt{4-r^2}}\,dr = \left| \begin{matrix} 4-r^2 = s \\ ds = -2r\,dr \\ r=0 \Rightarrow s=4 \\ r=1 \Rightarrow s=3 \end{matrix} \right| =$$

$$= \frac{3\pi}{4}\int_3^4 \frac{ds}{2\sqrt{s}} = \frac{3\pi}{4}\left[\sqrt{s}\right]_3^4 =$$

$$= \frac{3\pi}{4}\left(2-\sqrt{3}\right).$$

(b)



$$\iiint_K xy^2\,dV = \iint_D \left(\int_0^{1-x} xy^2\,dz\right)dx\,dy =$$

$$= \int_0^1 \int_0^1 xy^2(1-x)\,dx\,dy =$$

$$= \int_0^1 x(1-x)\,dx \int_0^1 y^2\,dy =$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{6}\cdot\frac{1}{3} = \frac{1}{18}$$

4. (a) $\begin{cases} \frac{\partial f}{\partial x} = y^2 - 4 = 0 \\ \frac{\partial f}{\partial y} = 2xy = 0 \end{cases} \Rightarrow \begin{cases} y = \pm 2 \\ x = 0 \end{cases}$

Vi har två kritiska punkter $(0, 2)$ och $(0, -2)$

$\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = 2y$, $\frac{\partial^2 f}{\partial y^2} = 2x$

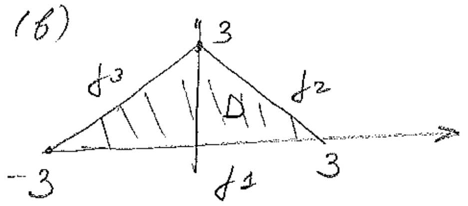
$$\mathcal{H}(0,2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,2) & \frac{\partial^2 f}{\partial x \partial y}(0,2) \\ \frac{\partial^2 f}{\partial x \partial y}(0,2) & \frac{\partial^2 f}{\partial y^2}(0,2) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$$

$\det \mathcal{H}(0,2) = -16 < 0 \Rightarrow (0,2)$ är en sadelpunkt

$$\mathcal{H}(0,-2) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

$\det \mathcal{H}(0,-2) = -16 < 0 \Rightarrow (0,-2)$ är en sadelpunkt.

(b)



Kritiska punkter inom D är $(0,2)$, $f(0,2) = 0$

Undersöker punkter på randen

γ_1: $(x, 0)$; $x \in [-3, 3]$

$f(x,0) = -4x$

$f_{max}\big|_{\gamma_1} = -4(-3) = 12$, $f_{min}\big|_{\gamma_1} = -4\cdot 3 = -12$

γ_2: $(x, 3-x)$, $x \in [0,3]$

$f(x, 3-x) = x(3-x)^2 - 4x$

låt $g(x) = x(3-x)^2 - 4x = x(x^2 - 6x + 5) = x(x-1)(x-5)$

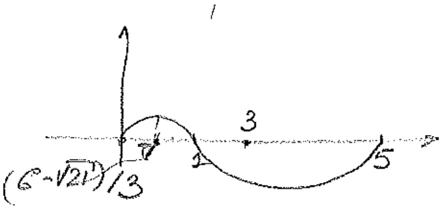
$g'(x) = 0 \Rightarrow (3-x)^2 - 2x(3-x) - 4 = 0$

$9 - 6x + x^2 - 6x + 2x^2 - 4 = 0$

$3x^2 - 12x + 5 = 0$

$$x = \frac{6 \pm \sqrt{36 - 15}}{3} = \frac{6 \pm \sqrt{21}}{3}$$

$\frac{6 + \sqrt{21}}{3} > 3$



$g(0) = 0$, $g(3) = -12$

$f_{max}\big|_{\gamma_2} = g_{max}\big|_{[0,3]} = g\left(\frac{6-\sqrt{21}}{3}\right)$

$f_{min}\big|_{\gamma_2} = g_{min}\big|_{[0,3]} = -12$

γ_3 : $(x, x+3)$, $x \in [-3, 0]$

$f(x, x+3) = x(x+3)^2 - 4x =$

$= x(x^2+6x+5) = x(x+1)(x+5)$

Låt $h(x) = x(x^2+6x+5) = x^3+6x^2+5x$

$h'(x) = 3x^2+12x+5 = 0$

$x = \frac{-6 \pm \sqrt{21}}{3}$,

$-\frac{6-\sqrt{21}}{3}$ ej inom intervallet $[-3, 0]$

$h(0) = 0$, $h(-3) = 12$

$f_{max}|_{\gamma_3} = h_{max}|_{[-3,0]} = 12$

$f_{min}|_{\gamma_3} = h_{min}|_{[-3,0]} = h\left(\frac{-6+\sqrt{21}}{3}\right)$

$g\left(\frac{6-\sqrt{21}}{3}\right) < 12$ och $h\left(-\frac{6+\sqrt{21}}{3}\right) > -12$.

Detta ger att $f_{max} = 12$ och $f_{min} = -12$.

5. (a) $\operatorname{curl} \mathbb{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y^2 & z^3 \end{vmatrix} = i \cdot 0 - j \cdot 0 + k \cdot 0 = 0$

Eftersom $\mathbb{R}^3$ är enkelt sammenhängande

blir $\mathbb{F}$ konservativt

(b) Man kan beräkna en potential φ till $\mathbb{F}$ och beräkna $\int_C \mathbb{F} \cdot d\mathbf{r}$ som

$\varphi(\mathbf{r}(\pi)) - \varphi(\mathbf{r}(0)) =$

$= \varphi(1, 1, 0) - \varphi(-1, -1, 0)$

$$\begin{cases} \frac{\partial \varphi}{\partial x} = x \\ \frac{\partial \varphi}{\partial y} = y^2 \\ \frac{\partial \varphi}{\partial z} = z^3 \end{cases}$$

Det är lätt att se att $\varphi(x, y, z) = \frac{x^2}{2} + \frac{y^3}{3} + \frac{z^4}{4}$ är en lösning

varav

$$\int_C \mathbb{F} \cdot d\mathbf{r} = \varphi(1, 1, 0) - \varphi(-1, -1, 0) = \frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = \frac{2}{3}$$

Alternativt kan man beräkna kurvintegralen m h a definition

$$\int_C \mathbb{F}\cdot d\mathbb{r} = \int_0^{\pi} \mathbb{F}(\mathbb{r}(t))\cdot \mathbb{r}'(t)\,dt =$$

$$= \int_0^{\pi} (-\cos t,\ \cos^2 t,\ 2\sqrt{2}\sin^3 t)\cdot(\sin t,\ \sin t,\ \sqrt{2}\cos t)\,dt$$

$$= \int_0^{\pi}\Big(-\cos t\sin t + \cos^2 t\sin t + 4\underbrace{\sin^3 t}_{(1-\cos^2 t)\sin t}\cos t\Big)dt =$$

$$= \left|\begin{array}{l}\cos t = s\\ ds = -\sin t\,dt\end{array}\right| =$$

$$= \int_{-1}^{1}\Big(-s + s^2 + 4(1-s^2)s\Big)ds = \Big[-\frac{s^2}{2} + \frac{s^3}{3} + 2s^2 - s^4\Big]_{-1}^{1} = \frac{2}{3}$$

6. Gör variabelbytet

$$\begin{cases} u = x + 2y \\ v = 2x + y \end{cases}$$

Området D avbildas på axelparallell rektangeln $0 \le u \le 1$, $-1 \le v \le 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 2 \\ 2 & 1\end{bmatrix} = 1 - 4 = -3 \quad \text{och} \quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$$

$$\iint_D (x+2y)\cos(2x+y)\,dx\,dy = \iint_{\substack{0\le u\le 1\\ -1\le v\le 1}} u\cos v\cdot\frac{1}{3}\,du\,dv =$$

$$= \frac{1}{3}\int_0^1 u\,du\int_{-1}^{1}\cos v\,dv = \frac{3^{-1}}{2}\cdot 2\sin 1 = 3^{-1}\sin 1$$

7. (a) Ytan kan parametriseras enligt

$$r(\varphi, z) = (2\cos\varphi,\ 2\sin\varphi,\ z), \quad \begin{array}{l} 0\le\varphi\le 2\pi \\ 1\le z\le 2\end{array}$$

En normal till ytan i punkten $(2\cos\varphi, 2\sin\varphi, z) = (x,y,z)$ är

$$\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial z} = \begin{vmatrix} i & j & k \\ -2\sin\varphi & 2\cos\varphi & 0 \\ 0 & 0 & 1\end{vmatrix} = 2\cos\varphi\, i + 2\sin\varphi\, j + 0\cdot k$$

$$= 2x\, i + 2y\, j \quad \text{(pekar utåt)}$$

(b) M h a parametriseringen i (a) får vi

$$\text{Flödet} = \iint_E F(r(\varphi,z))\cdot\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial z}\,d\varphi\,dz =$$

$$E:\ \begin{array}{l}0\le\varphi\le 2\pi\\ 1\le z\le 2\end{array}$$

$$= \iint_E \left(e^{4\cos^2\varphi} 2\sin\varphi\, i + \frac{\sin z}{z} j + e^{2\sin\varphi\, z} k\right) \cdot (2\cos\varphi\, i + 2\sin\varphi\, j + 0 k)\, d\varphi\, dz$$

$$= \iint_E \left(2e^{4\cos^2\varphi}\sin 2\varphi + 2\frac{\sin z}{z}\sin\varphi\right) d\varphi\, dz =$$

$$= \int_1^2 \underbrace{\left(\int_0^{2\pi} e^{4\cos^2\varphi}\sin 2\varphi\, d\varphi\right)}_{=0} dz + 2\int_1^3 \frac{\sin z}{z} \underbrace{\left(\int_0^{2\pi} \sin\varphi\, d\varphi\right)}_{=0} dz = 0$$