Chalmers tekniska högskola
Tentamen

Datum: 2018-08-20 kl. 14.00 - 18.00
Telefonvakt: Olof Giselsson

MVE500, TKSAM-2

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
För godkänt krävs 25 poäng på del 1. För betyget 4 krävs 35 poäng totalt, varav minst 6 poäng på del 2. För betyget 5 krävs 45 poäng totalt, varav minst 8 poäng på del 2. Varje godkänd dugga ger 1.5 bonuspoäng till del 1. Lösningar läggs ut på kursens hemsida. Resultat meddelas via Ladok.

Part 1 (mandatory exercises)

1. Determine rigorously whether the following sequences (a)–(b) and series (c) are divergent. (6p)

(a) $\left\{\dfrac{5n^3+2n^2+1}{n^3-1}\right\}_{n=1}^{\infty}$ (b) $\left\{\sin\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$ (c) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\cdot\cos(n)}{n\cdot 3^n}$

Solution: (a) Let $a_n = \frac{5n^3+2n^2+1}{n^3-1}$. Collecting n^3 above and below the fraction, we have that $a_n = \frac{5+2n^{-1}+n^{-3}}{1-n^{-3}}$. Therefore, since n^{-k} goes to 0, for $k>0$ when $n\to\infty$, using the properties of the limit a of sequence we have that $a_n \to 5$ when $n\to\infty$. Hence, the sequence is convergent to 5.

(b) Let $a_n = \sin\left(\frac{1}{n}\right)$. We know that n^{-1} goes to 0, when $n\to\infty$, and since $f(x)=\sin(x)$ is a continuous function, we have that:

$$\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n}\right) = \sin(0) = 0.$$

Hence, the sequence is convergent to 0.

(c) Let $a_n = \dfrac{2^n\cdot\cos(n)}{n\cdot 3^n}$. Then we have that

$$|a_n| = \frac{|2^n\cdot\cos(n)|}{|n\cdot 3^n|} \leq \frac{2^n}{n\cdot 3^n} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Let $b_n = \left(\dfrac{2}{3}\right)^n$. Then $\sum_{n=0}^{\infty} b_n$ is a geometric series and being $\dfrac{2}{3}<1$, it is convergent. By the comparison test and the previous inequality $\sum_{n=0}^{\infty}|a_n|$ is convergent too. Finally, for the absolute convergence test, we have that $\sum_{n=0}^{\infty} a_n$ is convergent.

2. (a) Determine the coefficients of the Maclaurin series of $f(x)=(a+bx)^n$, for $a,b>0$ and $n = 1,2,\ldots$. (3p)

(b) Determine the radius of convergence of the Maclaurin series of f. (2p)

Solution: (a) We notice that $f(x)=(a+bx)^n = a^n(1+\frac{b}{a}x)^n$. Using the known formulas for the binomial expansion we get that:

$$f(x) = \sum_{k=0}^{n} a^n\binom{n}{k}\left(\frac{b}{a}x\right)^k = \sum_{k=0}^{n} a^{n-k}b^k\binom{n}{k}x^k.$$

Hence, the Maclaurin coefficients are given by the formula $a_k = a^{n-k} b^k \binom{n}{k}$.

(b) Since the Mcalurin series of f is actually a finite sum, the radius of convergence is infinite.

3. (a) Find the length of the curve $\mathbf{r}(t) = \langle \frac{t^3}{3}, t^2, 2t \rangle$, for $t \in [0, 1]$. (3p)

(b) Find the curvature in $t = 1$ for the same curve. (3p)

Solution: (a) To find the length of the curve, we have to calculate $\dot{\mathbf{r}}(t)$. We have that:

$$\dot{\mathbf{r}}(t) = \langle t^2, 2t, 2 \rangle.$$

The length of the curve in $[0, 1]$ is given by the formula:

$$L = \int_0^1 |\dot{\mathbf{r}}(t)| dt = \int_0^1 (t^2 + 2) dt = \frac{7}{3}.$$

(b) To find the curvature of the curve $\kappa(t)$, in $t = 1$, we use the formula:

$$\kappa(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}.$$

We have that:

$$\ddot{\mathbf{r}}(t) = \langle 2t, 2, 0 \rangle.$$

Hence, we find that $\kappa(1) = \frac{2}{9}$.

4. (a) Find the linear approximation of $f(x, y) = \exp\left(\dfrac{x^2}{y-1}\right)$ at the point (1,-1). (3p)

(b) Sketch some of the level curves of $f(x, y) = 4x^2 + 9y^2$, for $-1 \leq f(x, y) \leq 1$. (3p)

Solution:

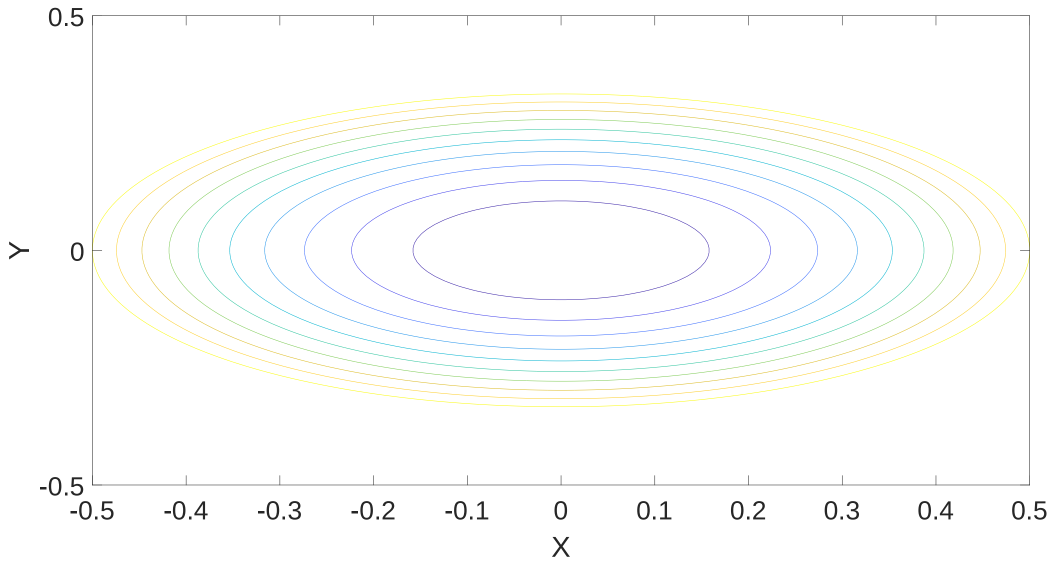
(a)The linear approximation in the point $(1, -1)$ is given by

$$L(x, y) = f(1, -1) + f_x(1, -1)(x - 1) + f_y(1, -1)(y + 1).$$

We have $f(1, -1) = \exp\left(-\frac{1}{2}\right)$, $f_x(1, -1) = -\exp\left(-\frac{1}{2}\right)$, $f_y(1, -1) = -\frac{1}{4}\exp\left(-\frac{1}{2}\right)$. Then,

$$L(x, y) = \exp\left(-\frac{1}{2}\right) \cdot \left(\frac{7}{4} - x - \frac{1}{4}y\right).$$

(b)



5. Find the stationary points of $f(x,y) = \arctan\big((x^2-1)y\big)$. When possible, use the second derivative test to classify them. (5p)

Solution: Stationary point means $\nabla f(x,y) = \mathbf{0}$, that gives the system of equations:

$$\begin{cases} \dfrac{2xy}{(y(x^2-1))^2+1} = 0 \\ \dfrac{(x^2-1)}{(y(x^2-1))^2+1} = 0 \end{cases} \iff \begin{cases} 2xy = 0 \\ x^2-1=0 \end{cases}$$

So we get as possible solutions:
$(\pm 1, 0)$.

To classify them we calculate the Hessian matrix:

$$H(x,y) = \frac{1}{(y(x^2-1))^2+1)^2} \cdot$$
$$\cdot \begin{bmatrix} 2y((y(x^2-1))^2+1) - 8x^2y^2(y(x^2-1)) & 2x((y(x^2-1))^2+1) - 4xy^2(x^2-1)^2 \\ 2x((y(x^2-1))^2+1) - 4xy^2(x^2-1)^2 & -2y(x^2-1)^3 \end{bmatrix}.$$

Therefore, we get:

$$H(\pm 1, 0) = \begin{bmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{bmatrix}.$$

Finally, we find that $\det(H(\pm 1,0)) = -1 < 0$. Hence, using the second derivative criterion, $(x,y) = (\pm 1, 0)$ are two saddle points.

6. Solve the heat equation: (6p)

$$\begin{cases} u_t(x,t) = 3u_{xx}(x,t) \quad \text{for } 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 \quad \text{for } t \geq 0 \\ u(x,0) = x(1-x) + \sin(\pi x) \quad \text{for } 0 \leq x \leq 1 \end{cases}$$

Solution: The heat equation is homogeneous with Dirichlet boundary conditions. From the theory we know that the solution has the following expression:

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n e^{-3n^2\pi^2 t} \right) \sin n\pi x.$$

The initial conditions determine a_n and b_n for all $n \geq 1$:

$$a_n = 2 \int_0^1 (x(1-x) + \sin(\pi x)) \sin(n\pi x)\, dx.$$

Evaluating the integrals, we get $a_1 = 1 + \frac{8}{\pi^3}$ and $a_n = 4\frac{1-(-1)^n}{(n\pi)^3}$ for $n > 1$.

Var god vänd blad!

Part 2 (Bonus exercises)

7. Let $F = 10 - x^2 - y^2 - z^2$ and for each real number a let $G_a = 3x + y + z - a$. For fixed a, maximize F subject to the constraint $G_a = 0$. (6p)

This maximal value varies with a, call it V_a. Also, let λ_a be the Lagrange multiplier for given a. Show by explicit calculations that $\frac{\partial V_a}{\partial a} = \lambda_a$.

Solution: Using the Lagrange multipliers we get the system:

$$\begin{cases} -2x = 3\lambda_a \\ -2y = \lambda_a \\ -2z = \lambda_a \end{cases}$$

From which we get $\lambda = -2y$, and the system:

$$\begin{cases} -2x & = 3y \\ y & = z \\ 3x + 2y & = a. \end{cases}$$

Solving for x, y, z, we get $x = 3/11a, y = z = 1/11a$ and $\lambda_a = -2/11a$.

We see that the maximum value is $F(3/11a, 1/11a, 1/11a) = 10 - 1/11a^2$. Finally, we get: $\frac{\partial V_a}{\partial a} = -2/11a = \lambda_a$.

8. Determine in two different ways the Maclaurin polynomial of order 6 of $f(x) = \sin(x)\cos(x)$. (5p)

Solution:

Let's denote with T_6 the Maclaurin polynomial of order 6. Avoiding the use of iterated derivatives, we consider the known formulas:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

Multiplying the two series and truncating the powers bigger than 6, we get:

$$T_6(x) = x - \frac{2x^3}{3} + \frac{2x^5}{15}.$$

Alternatively, we can use the known trigonometric identity: $2\sin(x)\cos(x) = \sin(2x)$ and the formula for the Maclaurin polynomial of the sin function to get:

$$T_6(x) = \frac{1}{2}\sin(2x) = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}\right) = x - \frac{2x^3}{3} + \frac{2x^5}{15}.$$

9. Solve the following inhomogeneous wave equation: (5p)

$$\begin{cases} u_{tt} = u_{xx} - 1, \qquad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, \quad u(1,t) = 1 \\ u(x,0) = x \\ u_t(x,0) = 0 \end{cases}$$

with inhomogeneous Dirichlet boundary conditions.

Solution: Since the heat equation is inhomogeneous, we take $u(x,t) = v(x,t) + s(x)$ in order to get v to satisfy a suitable homogeneous wave equation.
This gives

$$v_{tt} = v_{xx} + s'' - 1.$$

Hence we put $s''(x) = 1$, with boundary conditions $s(0) = 0$ and $s(1) = 1$. So we get:

$$s(x) = \frac{x^2 + x}{2}.$$

Then $v(x,t)$ satisfy the homogeneous wave equation:

$$\begin{cases} v_t = v_{xx}, \qquad 0 < x < 1, \quad t > 0 \\ v(0,t) = 0, \quad v(1,t) = 0 \\ v(x,0) = \frac{-x^2+x}{2} \\ v_t(x,0) = 0 \end{cases}$$

Therefore, by the theory we have the formula:

$$v(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x).$$

The initial conditions for v determine a_n and b_n, for any $n \geq 1$:

$$a_n = 2 \int_0^1 \frac{-x^2 + x}{2} \sin(n\pi x) dx = 2 \frac{(-1)^n - 1}{(n\pi)^3}$$

and

$$b_n = \frac{2}{n\pi} \int_0^1 0 \sin(n\pi x) dx = 0.$$

Finally, we get the expression for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} 2 \frac{(-1)^n - 1}{(n\pi)^3} \cos(n\pi t) \sin(n\pi x) + \frac{x^2 + x}{2}.$$

Lycka till!
Milo

Formelblad MVE500, HT-2016

Trigonometri

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \sin x \cos y = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$$

Integraler

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C, \quad a \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C \qquad \int \frac{1}{\sin^2 x}\,dx = -\cot x + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \neq 0 \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{\sqrt{a-x^2}}\,dx = \arcsin\frac{x}{\sqrt{a}} + C, \quad a > 0 \qquad \int \sqrt{a-x^2}\,dx = \frac{1}{2}x\sqrt{a-x^2} + \frac{a}{2}\arcsin\frac{x}{\sqrt{a}} + C, \quad a > 0$$

$$\int \frac{1}{\sqrt{a+x^2}}\,dx = \ln|x+\sqrt{x^2+a}| + C, \quad a \neq 0 \qquad \int \sqrt{a+x^2}\,dx = \frac{1}{2}\left(x\sqrt{a+x^2} + a\ln|x+\sqrt{x^2+a}|\right) + C$$

Maclaurinutvecklingar

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} + \dots \qquad |x| < 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k(k-1)\dots 1}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad -1 < x \leq 1$$

$$\arctan x = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad |x| \leq 1$$

Fourierserier

Jämn funktion $f(x) = f(-x)$
Udda funktion $f(x) = -f(-x)$

Fourierserien av en $2L$-periodisk funktion $f(x)$ ges av

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

där **Fourierkoefficienterna** ges av

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Den **komplex Fourierserien** av en $2L$-periodisk funktion $f(x)$ ges av

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

där de **komplexa Fourierkoefficienterna** ges av

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

Sinusserien av $f(x)$ definierad på intervallet $x \in [0, L]$ ges av

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,, \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Cosinusserien av $f(x)$ definierad på intervallet $x \in [0, L]$ ges av

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\,, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Parsevals identitet för en $2L$-periodisk funktion $f(x)$

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\,dx = \frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$