Chalmers tekniska högskola | Datum: 2018-10-29 kl. 14.00 - 18.00
Tentamen | Telefonvakt: Milo Viviani

MVE500, TKSAM-2

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
För godkänt krävs 25 poäng totalt. För betyget 4 krävs 35 poäng totalt. För betyget 5 krävs 45 poäng totalt. Varje godkänd dugga ger 1.5 bonuspoäng. Lösningar läggs ut på kursens hemsida. Resultat meddelas via Ladok.

Part 1 (mandatory exercises)

1. Determine whether and accordingly to which criterion the following series are convergent or divergent. (6p)

(a) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ (b) $\sum_{n=1}^{\infty} \frac{3^n n^n}{(2n+1)^n}$ (c) $\sum_{n=0}^{\infty} \frac{n \cdot \arctan(n)}{(n+1)\cdot(1.5)^n}$

Solution: (a) Let $a_n = \frac{\cos(n\pi)}{n}$. Since $\cos(n\pi) = (-1)^n$, then $a_n = \frac{(-1)^n}{n}$. Therefore the series is convergent by the alternating series test.

(b) Let $a_n = \frac{3^n n^n}{(2n+1)^n}$. Then $\sqrt[n]{a_n} = \frac{3n}{2n+1} \to \frac{3}{2} > 1$, for $n \to \infty$. Therefore, by the root test, the series is divergent.

(c) Let $a_n = \frac{n \cdot \arctan(n)}{(n+1)\cdot(1.5)^n} = \frac{n \cdot \arctan(n)}{n+1}\frac{2^n}{3^n}$ and let $b_n = \frac{2^n}{3^n}$.

Then we have that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n \cdot \arctan(n)}{n+1} = \frac{\pi}{2} > 0.$$

Moreover, $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2^n}{3^n}$ converges because it is a geometric series. Finally, by the limit comparison test, we can conclude that $\sum_{n=0}^{\infty} a_n$ converges too.

2. (a) Determine the Maclaurin polynomial of order 5 of $f(x) = x^2 \sin x \cos x$. (3p)

(b) Determine the radius of convergence of the Maclaurin series of $g(x) = x\sqrt{3+2x}$, using the formulas provided at the end of the exam sheet. (2p)

Solution: (a) We notice that $f(x) = \frac{x^2}{2}\sin 2x$. Using the known formulas for the Mclaurin series of $\sin x$, we get that:

$$f(x) = \frac{x^2}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} 2^{2k}\frac{(-1)^k x^{2k+3}}{(2k+1)!}.$$

Hence, the Maclaurin polynomial of order 5 is given by the formula $T_5(x) = x^3 - 2\frac{x^5}{3}$.

(b) We notice that $g(x) = x\sqrt{3}(1+\frac{2}{3}x)^{1/2}$. Looking at the radius of convergence for the Mclaurin series of $(1+x)^\alpha$, we must have $|\frac{2}{3}x| < 1$. Hence, $|x| < \frac{3}{2}$. Therfore the radius of convergence is $R = \frac{3}{2}$.

3. (a) Determine and parametrize by $t \in \mathbb{R}$ the curve $\mathbf{r} = \mathbf{r}(t)$, obtained by intersecting the ellipsoid $x^2 + y^2 + 9z^2 = 25$ with the plane $z = 1$. (3p)

(b) Find the curvature at the point $(4, 0, 1)$, for the curve $\mathbf{r} = \mathbf{r}(t)$ found in (a). (3p)

Solution: (a) The intersection of the two surfaces give the space curve $C = \{(x, y, z) | x^2 + y^2 = 16,\ z = 1\}$, which can be parametrized by $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 1\rangle$.

(b) We first notice that $\mathbf{r}(t) = \langle 4, 0, 1\rangle$ when $t = 0$. To find the curvature of the curve $\kappa(t)$, in $t = 0$, we use the formula:

$$\kappa(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}.$$

We have that: $\dot{\mathbf{r}}(t) = 4\langle -\sin t, \cos t, 0\rangle$, $\ddot{\mathbf{r}}(t) = -4\langle \cos t, \sin t, 0\rangle$.

Hence, we find that $\kappa(0) = \frac{1}{4}$.

One could also notice that the curve is nothing else than a circle with radius $R = 4$ and therefore its curvature has to be constantly equal to $\kappa(0) = \frac{1}{4}$.

4. (a) Find the tangent plane at the graph of $f(x, y) = 2\cos x \sin y$ at the point $(\pi, \pi/2, -2)$. (3p)

(b) Sketch the level curve $L_0 = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$. (3p)

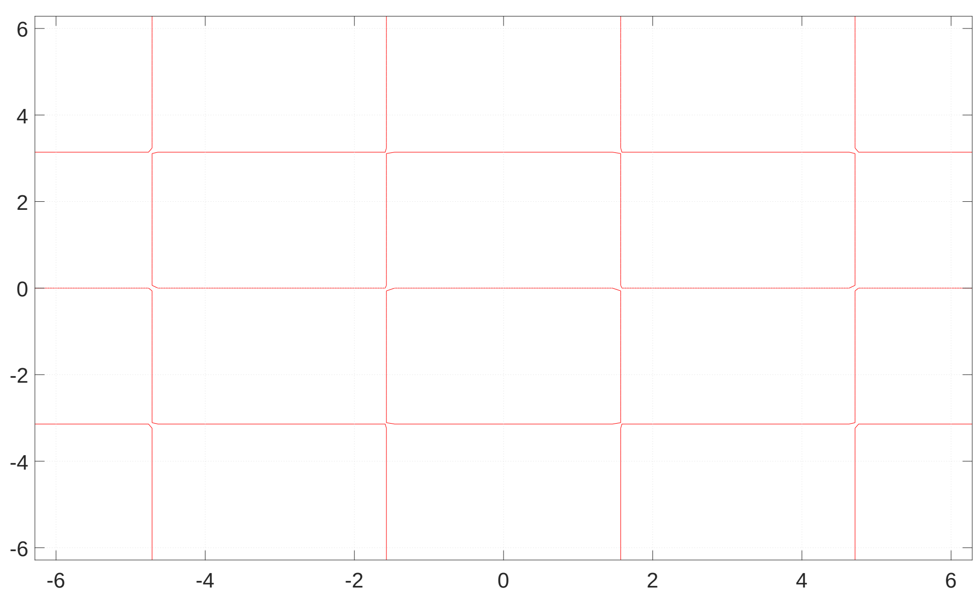
Solution:

(a)The tangent plane at the point $(\pi, \pi/2, -2)$ is given by

$$z + 2 = f_x(\pi, \pi/2)(x - \pi) + f_y(\pi, \pi/2)(y - \pi/2).$$

We have $f_x(\pi, \pi/2) = 0$, $f_y(\pi, \pi/2) = 0$. Then, the tangent plane is given by $z = -2$.

(b) The set L_0 is determined by either $\cos x = 0$ or $\sin y = 0$ and coincides with the red locus in the figure below.



5. Find the stationary points of $f(x, y) = \exp(5 + \ln(x^2 - x(y+1) + y^2 + 1))$. When possible, use the second derivative test to classify them. (5p)

Solution: We first notice that:

$$f(x, y) = \exp(5)(x^2 - x(y+1) + y^2 + 1).$$

Stationary point means $\nabla f(x, y) = \mathbf{0}$, that gives the system of equations:

$$\begin{cases} \exp(5)(2x - (y+1)) = 0 \\ \exp(5)(-x + 2y) = 0 \end{cases} \iff \begin{cases} x = \dfrac{y+1}{2} \\ x = 2y \end{cases} \iff \begin{cases} y = \dfrac{1}{3} \\ x = \dfrac{2}{3} \end{cases}$$

So we get as possible solution $\frac{1}{3}(2, 1)$. We also notice that in this point $x^2 - x(y+1) + y^2 + 1$ is positive, and therefore the function is defined there.

To classify them we calculate the Hessian matrix:

$$H(x, y) = \exp(5)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Therefore, we get:

$$H\left(\frac{2}{3}, \frac{1}{3}\right) = \exp(5)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Finally, we find that $\det(H\left(\frac{2}{3}, \frac{1}{3}\right)) = 3\exp(10) > 0$ and $2\exp(10) > 0$. Hence, using the second derivative criterion, $(x, y) = \left(\frac{2}{3}, \frac{1}{3}\right)$ is a minimum point.

6. Determine the sine-series of $f(x) = x^2 - x\pi$, for $0 \le x \le \pi$. (4p)

Then, use the obtained result to solve the heat equation : (2p)

$$\begin{cases} u_t(x, t) = 2u_{xx}(x, t) & \text{for } 0 < x < \pi, t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t \ge 0 \\ u(x, 0) = x^2 - x\pi & \text{for } 0 \le x \le \pi \end{cases}$$

Solution: The sine-series for $f(x) = x^2 - x\pi$, for $0 \le x \le \pi$, is given by:

$$f(x, t) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(x - \pi)\sin(nx)dx = \frac{4}{\pi n^3}((-1)^n - 1).$$

The heat equation is homogeneous with Dirichlet boundary conditions. From the theory we know that the solution has the following expression:

$$u(x, t) = \sum_{n=1}^{\infty}\left(b_n e^{-2n^2 t}\right)\sin nx.$$

Var god vänd blad!

Part 2 (Bonus exercises)

7. State and prove the *limit comparison test.* (5p)

Solution: See J. Stewart, Calculus: Early Transcendentals, 8:e upplagan, Metric Edition, pag. 729.

8. What point on the sphere $x^2 + y^2 + z^2 = 1$ is closest to the point $x = 1$, $y = 2$, $z = 3$? (5p)

Solution:

Let's call $d(Q, P) = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$ the distance between a point $Q = (x, y, z)$ to $P = (1, 2, 3)$. For our purpose, it is equivalent to consider $f(x, y, z) := d(Q, P)^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$. The constraint is then given by $g(x, y, z) = 0$, for $g(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then, the Lagrange multipliers technique provides the following system of equations to be solved:

$$\begin{cases} x - 1 = \lambda x \\ y - 2 = \lambda y \\ z - 3 = \lambda z \\ x^2 + y^2 + z^2 = 1. \end{cases}$$

This gives the solution $P' := \frac{1}{\sqrt{14}}(1, 2, 3)$, which is the closest point to P lying on the unit sphere.

9. Solve the following wave equation, using the method of separation of variables: (6p)

$$\begin{cases} u_{tt} = u_{xx} + u, \qquad 0 < x < 1, \quad t > 0 \\ u(0, t) = 0, \quad u(1, t) = 0 \\ u(x, 0) = -5\sin(3\pi x) \\ u_t(x, 0) = x - 1. \end{cases}$$

Solution: Using the separation of variables $u(x, t) = X(x)T(t)$, and substituting it into the wave equation, we get:

$$\begin{cases} X''(x) + \mu^2 X(x) = 0 \\ T''(t) + (\mu^2 - 1)T(t) = 0, \end{cases}$$

for μ constant. This gives:

$$\begin{cases} T(t) = a\cos(\sqrt{\mu^2 - 1}t) + b\sin(\sqrt{\mu^2 - 1}t) \\ X(x) = c\cos(\mu x) + d\sin(\mu x). \end{cases}$$

Using the boundary conditions, we can conclude that it must be $d = 0$ and $\mu = \pm n\pi$, for $n = 1, 2, 3, ...$ Finally, using the superposition principle, we get:

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(\sqrt{n^2\pi^2 - 1}t) + b_n \sin(\sqrt{n^2\pi^2 - 1}t))\sin(n\pi x).$$

The initial conditions for u determine a_n and b_n, for any $n \geq 1$:

$$a_n = 2 \int_0^1 -5 \sin 3\pi x \sin(n\pi x) dx$$

and

$$b_n = \frac{2}{\sqrt{n^2\pi^2 - 1}} \int_0^1 (x-1) \sin(n\pi x) dx.$$

The integral above give: $a_n = 0$ for $n \neq 3$ and $a_3 = -5$, $b_n = -\frac{2}{\sqrt{n^2\pi^2-1}} \frac{1}{n\pi}$.

Lycka till!
Milo

Formelblad MVE500, HT-2018

Trigonometri

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \sin x \cos y = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$$

Integraler

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C, \quad a \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C \qquad \int \frac{1}{\sin^2 x}\,dx = -\cot x + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \neq 0 \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{\sqrt{a-x^2}}\,dx = \arcsin\frac{x}{\sqrt{a}} + C, \quad a > 0 \qquad \int \sqrt{a-x^2}\,dx = \frac{1}{2}x\sqrt{a-x^2} + \frac{a}{2}\arcsin\frac{x}{\sqrt{a}} + C, \quad a > 0$$

$$\int \frac{1}{\sqrt{a+x^2}}\,dx = \ln|x + \sqrt{x^2+a}| + C, \quad a \neq 0 \qquad \int \sqrt{a+x^2}\,dx = \frac{1}{2}\left(x\sqrt{a+x^2} + a\ln|x+\sqrt{x^2+a}|\right) + C$$

Maclaurinutvecklingar

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} + \dots \qquad |x| < 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k(k-1)\dots 1}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad -1 < x \leq 1$$

$$\arctan x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad |x| \leq 1$$

Fourierserier

Jämn funktion $f(x) = f(-x)$
Udda funktion $f(x) = -f(-x)$

Fourierserien av en $2L$-periodisk funktion $f(x)$ ges av

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

där **Fourierkoefficienterna** ges av

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Den **komplex Fourierserien** av en $2L$-periodisk funktion $f(x)$ ges av

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

där de **komplexa Fourierkoefficienterna** ges av

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

Sinusserien av $f(x)$ definierad på intervallet $x \in [0, L]$ ges av

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,, \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Cosinusserien av $f(x)$ definierad på intervallet $x \in [0, L]$ ges av

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\,, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Parsevals identitet för en $2L$-periodisk funktion $f(x)$

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\,dx = \frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$