Chalmers tekniska högskola — Datum: 2019-01-07 kl. 08.30–12.30
Tentamen — Telefonvakt: Linnea Hietala

MVE500

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
För godkänt krävs 25 poäng totalt. För betyget 4 krävs 35 poäng totalt. För betyget 5 krävs 45 poäng totalt. Varje godkänd dugga ger 1.5 bonuspoäng. Lösningar läggs ut på kursens hemsida. Resultat meddelas via Ladok.

Part 1 (mandatory exercises)

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$ is convergent or divergent for: (6p)

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$

and, when convergent, determine its sum.

Solution: (a) For $x = 1$, the above series is convergent by the alternating series test. Moreover, for $x = 1$ it coincides with the Maclaurin series of $\ln(1 + x)$. Hence it converges to $\ln(1 + 1) = \ln(2)$.

(b) For $x = -1$, the above series is divergent, being minus the harmonic series, which is divergent.

(c) For $x = 0$, the above series is convergent, being every term equals to zero. Therefore, the series is convergent to 0.

2. (a) Determine the Maclaurin series of $f(x) = \cos(3x^2 + \frac{\pi}{2})$. (3p)

(b) Determine the radius of convergence of the Maclaurin series of $g(x) = \ln(5+\frac{5}{2}x)$, using the formulas provided at the end of the exam sheet. (2p)

Solution: (a) Using the identity $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$ and the known formulas for the Maclaurin series of $\sin x$, we get that:

$$f(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k(3x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}3^{2k+1}x^{4k+2}}{(2k+1)!}.$$

(b) We notice that $g(x) = \ln(5(1 + \frac{x}{2})) = \ln(5) + \ln(1 + \frac{x}{2})$. Looking at the radius of convergence for the Maclaurin series of $\ln(1 + x)$, we must have $|\frac{x}{2}| < 1$. Hence, $|x| < 2$. Therfore the radius of convergence is $R = 2$.

3. (a) Determine the length of the curve $\mathbf{r}(t) = \langle t, \frac{1}{6}t^3 + \frac{1}{2}\frac{1}{t}, 0\rangle$, from $t = 1$ to $t = 2$. (3p)

(b) Find the curvature at $t = 1$, for the curve in (a). (3p)

Solution: (a) To find the length of the curve, we have to calculate $\dot{\mathbf{r}}(t)$. We have that:

$$\dot{\mathbf{r}}(t) = \langle 1, \frac{1}{2}t^2 - \frac{1}{2}\frac{1}{t^2}, 0\rangle.$$

The length of the curve in $[1, 2]$ is given by the formula:

$$L = \int_1^2 |\dot{\mathbf{r}}(t)|dt = \int_1^2 \sqrt{1 + \frac{1}{4}t^4 - \frac{1}{2} + \frac{1}{4}\frac{1}{t^4}}dt = \int_1^2 \frac{1}{2}t^2 + \frac{1}{2}\frac{1}{t^2}dt = \frac{17}{12}.$$

(b) To find the curvature of the curve $\kappa(t)$, in $t = 1$, we use the formula:

$$\kappa(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}.$$

We have that: $\ddot{\mathbf{r}}(t) = \langle 0, t + \frac{1}{t^3}, 0 \rangle$. Hence, we find that $\kappa(1) = 2$.

4. (a) Find the tangent plane at the graph of $f(x,y) = \ln(xy)$ at the point $(5, 1/5, 0)$. (3p)

(b) Sketch the level curves $L_k = \{(x,y) \in \mathbb{R}^2 | f(x,y) = k\}$, for $f(x,y)$ as in (a) and $k = 0, 1, 2, 3$. (3p)

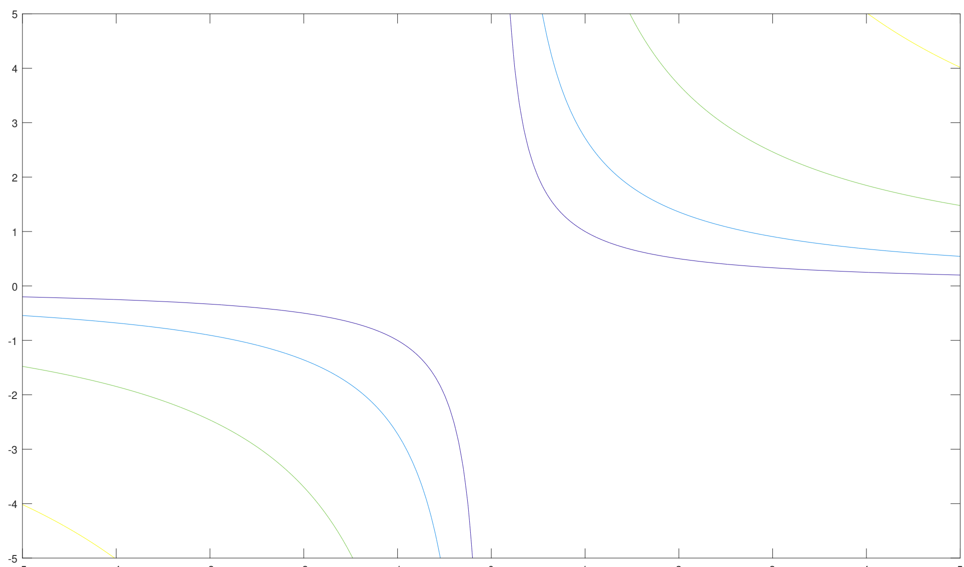
Solution:

(a)The tangent plane at the point $(5, 1/5, 0)$ is given by

$$z = f_x(5, 1/5)(x - 5) + f_y(5, 1/5)(y - 1/5).$$

Noticing that $f(x,y) = \ln(xy) = \ln(x) + \ln(y)$, we have $f_x(5, 1/5) = 1/5$, $f_y(5, 1/5) = 5$. Then, the tangent plane is given by $z = 1/5x + 5y - 2$.

(b) The sets L_k, for $k = 0, 1, 2, 3$, are given in the figure below.



5. Find the stationary points of $f(x,y) = x + y + \frac{1}{xy}$. When possible, use the second derivative test to classify them. (5p)

Solution: Stationary point means $\nabla f(x,y) = \mathbf{0}$, that gives the system of equations:

$$\begin{cases} 1 - \dfrac{1}{x^2 y} = 0 \\ 1 - \dfrac{1}{x y^2} = 0 \end{cases} \iff \begin{cases} x = y \\ x = 1 \end{cases}$$

So we get as possible solution $(1, 1)$.

To classify it, we calculate the Hessian matrix:

$$H(x,y) = \begin{bmatrix} \frac{2}{x^3y} & \frac{1}{x^2y^2} \\ \frac{1}{x^2y^2} & \frac{2}{xy^3} \end{bmatrix}.$$

Therefore, we get:

$$H(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Finally, we find that $\det(H(1,1)) = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$. Hence, using the second derivative criterion, $(x,y) = (1,1)$ is a minimum point.

6. Determine the sine-series of

$$f(x) = \begin{cases} 0 \text{ for } 0 < x \leq 1 \\ x-1 \text{ for } 1 < x \leq 2. \end{cases}$$

. (4p)

Then, use the obtained result to solve the heat equation : (2p)

$$\begin{cases} u_t(x,t) = 5u_{xx}(x,t) & \text{for } 0 < x < 2, t > 0 \\ u(0,t) = u(2,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & \text{for } 0 \leq x \leq 2 \end{cases}$$

Solution: The sine-series for $f(x)$, for $0 \leq x \leq 2$, is given by:

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x,$$

where:

$$b_n = \int_1^2 (x-1) \sin\left(\frac{n\pi}{2} x\right) dx = \begin{cases} \frac{2}{n\pi} - \frac{4}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \text{ for } n = 1,3,5,\ldots \\ -\frac{2}{n\pi} \text{ for } n = 0,2,4,\ldots. \end{cases}$$

The heat equation is homogeneous with Dirichlet boundary conditions. From the theory we know that the solution has the following expression:

$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n e^{-5\frac{n^2\pi^2}{4}t}\right) \sin \frac{n\pi}{2} x.$$

Var god vänd blad!

Part 2 (Bonus exercises)

7. State and prove the necessary condition for a series to be convergent, also known as *divergence test*. (5p)

Solution: See J. Stewart, Calculus: Early Transcendentals, 8:e upplagan, Metric Edition, pag. 713.

8. Find the maximum and minimum values of the function $f(x, y) = x^2 - x/2 + y^2 - y$ on the disc enclosed by the unit circle $x^2 + y^2 = 1$ (consider both the interior disc and its boundary). (5p)

Solution:

There is one interior critical point at $(1/4, 1/2)$, which is the minimum. Using Lagrange multipliers, there are two critical points on the bound circle, namely $(\sqrt{5}/5, 2\sqrt{5})$ and $(-\sqrt{5}/5, -2\sqrt{5})$. The second of these is the global maximum.

9. Solve the inhomogeneous wave equation: (6p)

$$\begin{cases} u_{tt} = 4u_{xx} + 4\sin x, \qquad 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0, \quad u(\pi, t) = 0 \\ u(x, 0) = 0 \\ u_t(x, 0) = 0. \end{cases}$$

Solution: The solution of an inhomogeneous wave equation can be reduced to the solution of an homogeneous one by using the splitting $u(x,t) = v(x,t) + s(x)$. Hence, we get:

$$\begin{cases} v_{tt} = 4v_{xx}, \qquad 0 < x < \pi, \quad t > 0 \\ v(0,t) = 0, \quad v(\pi, t) = 0 \\ v(x, 0) = -\sin x \\ v_t(x, 0) = 0. \end{cases}$$

and $s(x) = \sin x$.

Then we have:

$$v(x,t) = \sum_{n=1}^{\infty} (a_n \cos(2nt) + b_n \sin(2nt)) \sin(nx).$$

The initial conditions for u determine a_n and b_n, for any $n \geq 1$:

$$a_n = \frac{2}{\pi} \int_0^{\pi} -\sin x \sin(nx) dx$$

and

$$b_n = \frac{1}{n\pi} \int_0^{\pi} 0 \sin(nx) dx.$$

The integral above give: $a_1 = -1$ and $a_n = 0$ for $n \neq 1$, $b_n = 0$.

Finally, we get $u(x,t) = -\cos(2t)\sin x + \sin x$.

Lycka till!
Milo