Chalmers tekniska högskola | Datum: 2019-08-19 kl. 14.00 - 18.00
Tentamen | Telefonvakt: Juan Inda

MVE500

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
För godkänt krävs 25 poäng totalt. För betyget 4 krävs 35 poäng totalt. För betyget 5 krävs 45 poäng totalt. Varje godkänd dugga ger 1.5 bonuspoäng. Lösningar läggs ut på kursens hemsida. Resultat meddelas via Ladok.

Part 1 (mandatory exercises)

1. Determine whether the following series are convergent or divergent: (6p)

(a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{\log(n+1)}$ (c) $\sum_{n=0}^{\infty} \frac{2^{2n}}{7^n}$

and, when convergent, determine its sum.

Solution: (a) $\frac{(-2)^n}{n}$ does not converge to 0, for $n \to \infty$. Therefore, by the divergence test the series is divergent.

(b) $\log(n+1) < n+1$, for every $n \geq 1$. Therefore, $\frac{1}{\log(n+1)} > \frac{1}{n+1}$. Hence, by the comparison test, the series is divergent.

(c) $\frac{2^{2n}}{7^n} = \left(\frac{4}{7}\right)^n$ and $\frac{4}{7} < 1$. Hence, the series is a geometric series convergent to $\frac{7}{3}$.

2. (a) Determine the Maclaurin series of $f(x) = \cos(2x - \frac{\pi}{6})\cos(\frac{\pi}{6}) - \sin(2x - \frac{\pi}{6})\sin(\frac{\pi}{6})$. [Hint: use the trigonometric formulas.] (3p)

(b) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{5x^{2n}}{n!}$. (2p)

Solution: (a) Using the identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, we get that $f(x) = \cos(2x)$. Hence, its McLaurin series is $\sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n}}{(2n)!}$.

(b) We notice that the power series represents the function $g(x) = 5e^{x^2}$, for every $x \in \mathbb{R}$, being the Mclaurin series of the exponential convergent for any $x \in \mathbb{R}$. Therefore the radius of convergence is $R = \infty$.

3. (a) Determine the tangent vector $\mathbb{T}(t)$ at $t = \pi$ and the length of the curve $\mathbf{r}(t) = \langle -\cos(\frac{\pi}{3})\cos(t), -\sin(\frac{\pi}{3})\cos(t), \sin(t)\rangle$, from $t = 1$ to $t = 2$. (3p)

(b) Find the curvature at $t = \pi$, for the curve in (a). (3p)

Solution: (a) We first calculate $\dot{\mathbf{r}}(t)$. We have that:

$$\dot{\mathbf{r}}(t) = \langle \cos(\frac{\pi}{3})\sin(t), \sin(\frac{\pi}{3})\sin(t), \cos(t)\rangle.$$

Them, we see that $|\dot{\mathbf{r}}(t)| = 1$, for every $t \in \mathbb{R}$. Therefore, $\mathbb{T}(\pi) = \dot{\mathbf{r}}(\pi) = (0, 0, -1)$ and the length of the curve in $[1, 2]$ is $L = 1$.

(b) To find the curvature of the curve $\kappa(t)$, in $t = \pi$, we use the formula:

$$\kappa(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}.$$

We have that: $\ddot{\mathbf{r}}(t) = \langle \cos(\frac{\pi}{3})\cos(t), \sin(\frac{\pi}{3})\cos(t), -\sin(t)\rangle$. Hence, we find that $\kappa(\pi) = 1$.

4. (a) Find the tangent plane at the graph of $f(x, y) = \ln(x^3) - \ln(y^3)$ at the point $(1, 1, 0)$. (3p)

(b) Sketch the level curves $L_k = \{(x, y) \in \mathbb{R}^2 | f(x, y) = k\}$, for $f(x, y)$ as in (a) and $k = 0, 1, 2, 3$. (3p)

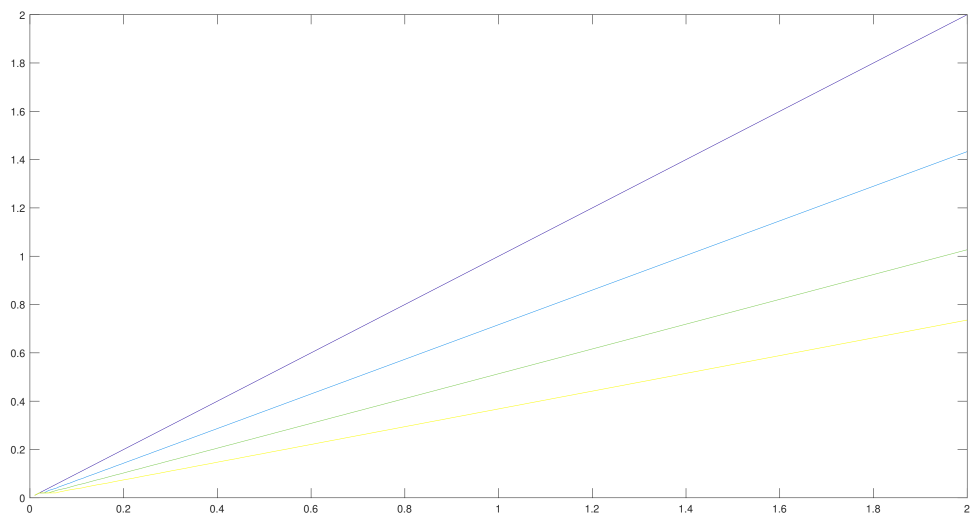
Solution:

(a)The tangent plane at the point $(1, 1, 0)$ is given by

$$z = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1).$$

We have $f_x(1, 1) = 3$, $f_y(1, 1) = -3$. Then, the tangent plane is given by $z = 3x + 3y$.

(b) The sets L_k, for $k = 0, 1, 2, 3$, are given in the figure below. These can be found noticing that $f(x, y) = \ln((\frac{x}{y})^3)$. Hence, $f(x, y) = k$ is given by $\frac{x}{y} = e^{k/3}$, for $x > 0, y > 0$. These are straights lines through the origin, with the point $(0, 0)$ removed for any $k \in \mathbb{R}$.



5. Find the stationary points of $f(x, y) = y\sin(x) - \cos(y)$ in the domain $D = \{-\pi < x < \pi, -\pi < y < \pi\}$. When possible, use the second derivative test to classify them. (5p)

Solution: Stationary point means $\nabla f(x, y) = \mathbf{0}$, that gives the system of equations:

$$\begin{cases} y\cos(x) = 0 \\ \sin(x) + \sin(y) = 0 \end{cases} \iff \begin{cases} y = 0 \\ x = 0 \end{cases} \text{or} \begin{cases} x = \pm\dfrac{\pi}{2} \\ y = \mp\dfrac{\pi}{2} \end{cases}$$

So we get as possible solutions $(0, 0), (\pm\frac{\pi}{2}, \mp\frac{\pi}{2})$.

To classify it, we calculate the Hessian matrix:

$$H(x,y) = \begin{bmatrix} -y\sin(x) & \cos(x) \\ \cos(x) & \cos(y) \end{bmatrix}.$$

For $(x,y) = (0,0)$, we get:

$$H(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $\det(H(0,0)) = -1 < 0$, the point is a saddle point. For $(x,y) = (\pm\frac{\pi}{2}, \mp\frac{\pi}{2})$, we get:

$$H(\pm\frac{\pi}{2}, \mp\frac{\pi}{2}) = \begin{bmatrix} -\frac{\pi}{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\det(H(\pm\frac{\pi}{2}, \mp\frac{\pi}{2})) = 0$, the criterion is inconclusive.

6. Determine the sine-series of

$$f(x) = \cos(x),$$

for $x \in [0,\pi]$. Then, use the obtained result to solve the heat equation : (4p)

(2p)

$$\begin{cases} u_t(x,t) = 2u_{xx}(x,t) & \text{for } 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & \text{for } 0 \leq x \leq \pi \end{cases}$$

Solution: The sine-series for $f(x)$, for $0 \leq x \leq \pi$, is given by:

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)\,dx = \begin{cases} 0 \text{ for } n = 1,3,5,\dots \\ \dfrac{4n}{\pi(n^2-1)} \text{ for } n = 2,4,\dots. \end{cases}$$

The heat equation is homogeneous with Dirichlet boundary conditions. From the theory we know that the solution has the following expression:

$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n e^{-2n^2 t}\right)\sin(nx).$$

Var god vänd blad!

Part 2 (Bonus exercises)

7. State and prove the theorem saying that the gradient is the direction of maximal growth for the directional derivative of a differentiable function. (5p)

Solution: See J. Stewart, Calculus: Early Transcendentals, 8:e upplagan, Metric Edition, theorem 15 pag. 952.

8. Find the maximum and minimum values of the function $f(x, y) = \exp(x^2 + y^2)$ in the square $Q = \{(x, y)| -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}}\}$ (consider both the interior square and its boundary). (5p)

Solution:

There is one interior critical point at $(0, 0)$, which is the minimum. The maximum has to be on the boundary and there are four points where is attained, i.e. $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$,$(\mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. Note that these points cannot be found using Lagrange multipliers, which in this case can only determine the minima of the function restricted to the boundary.

9. Solve the inhomogeneous heat equation: (6p)

$$\begin{cases} u_t = u_{xx} + \sin(\frac{\pi}{2}x), \qquad 0 < x < 2, \quad t > 0 \\ u(0, t) = 0, \quad u(2, t) = 2 \\ u(x, 0) = x. \end{cases}$$

Solution: The solution of an inhomogeneous heat equation can be reduced to the solution of an homogeneous one by using the splitting $u(x, t) = v(x, t) + s(x)$. Hence, we get:

$$\begin{cases} v_t = v_{xx}, \qquad 0 < x < \pi, \quad t > 0 \\ v(0, t) = 0, \quad v(\pi, t) = 0 \\ v(x, 0) = -\frac{4}{n\pi}\sin(\frac{\pi}{2}x), \end{cases}$$

and $s(x) = \frac{4}{\pi^2}\sin(\frac{\pi}{2}x) + x$.

Then we have:

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{\pi^2 n^2}{4}t}\sin(\frac{n\pi}{2}x).$$

The initial condition for v determines b_n, for any $n \ge 1$:

$$b_n = \int_0^2 \frac{4}{\pi^2}\sin(\frac{\pi}{2}x)\sin(\frac{n\pi}{2}x)dx.$$

The integral above give: $b_1 = -\frac{4}{\pi^2}$ and $b_n = 0$ for $n \neq 2$.

Finally, we can get $u(x, t) = -\frac{4}{\pi^2}e^{-\frac{\pi^2}{4}t}\sin(\frac{\pi}{2}x) + \frac{4}{\pi^2}\sin(\frac{\pi}{2}x) + x$, summing up $v(x, t)$ and $s(x)$.

Lycka till!
Milo