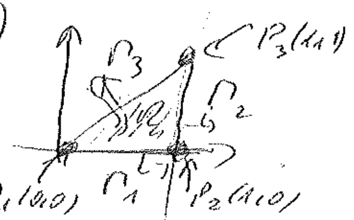
Bonus Problems 5 solutions (1)



5.1 (a) $\lambda(x,y) = 1+x+y \qquad f(x,y) = 1 \qquad u_A = 5$

The equation:

$$-\nabla\cdot\big((1+x+y)\nabla u(x,y)\big) = -\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)\left((1+x+y)\frac{\partial u}{\partial x}\vec{i}+(1+x+y)\frac{\partial u}{\partial y}\vec{j}\right)$$

$$= \boxed{(1+x+y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 1}$$

Boundary:

Γ_1: $\gamma = 2 \qquad \vec{n} = -\vec{j} \qquad y = 0 \quad 0<x<1$

$$\partial_N u = \left(\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}\right)\cdot(-\vec{j}) = -\frac{\partial u}{\partial y}$$

$$\lambda\,\partial_N u + \gamma(u-u_A) = \boxed{-(1+x)\frac{\partial u}{\partial y}(x,0) + 2(u(x,0)-5) = 0}$$

Γ_2: $\gamma = 1 \qquad \vec{n} = \vec{i} \qquad x = 1 \quad 0<y<1$

$$\partial_N u = \left(\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}\right)\cdot\vec{i} = \frac{\partial u}{\partial x}$$

$$\lambda\,\partial_N u + \gamma(u-u_A) = (2+y)\frac{\partial u}{\partial x}(1,y) + 1\cdot(u(1,y)-5) = 0$$

Γ_3: $\gamma = 0 \qquad \vec{n} = \frac{1}{\sqrt{2}}(-\vec{i}+\vec{j}) \qquad x = x \quad y = x \quad 0<x<1$

$$\partial_N u = \left(\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}\right)\left(-\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}\right) = -\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x}+\frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}$$

$$(1+2x)\left(-\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x}(x,x)+\frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}\right) = 0 \implies \frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(x,x).$$

Boundary value problem:

Find $u = u(x,y)$ such that

$$\begin{cases}
(1+x+y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 1 & (x,y)\in R \quad 0.2\\
-(1+x)\frac{\partial u}{\partial y}(x,0) + 2(u(x,0)-5) = 0 & 0<x<1 \quad 0.2\\
(2+y)\frac{\partial u}{\partial x}(1,y) + u(1,y)-5 = 0 & 0<y<1 \quad 0.2\\
\frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(x,x) & 0<x<1 \quad 0.2
\end{cases}$$

$\Sigma = 0.8$

(b) General weak formulation:

$$\iint_\Omega \nabla u\cdot\nabla v\,dA + \int_\Gamma k\,u\,v\,ds = \iint_\Omega f v\,dA + \int_\Gamma (g+k u_A)\,v\,ds$$

Ω: $0<x<1$ $\quad 0<y<x$

$$\int_{\Gamma_1} h\,ds = \left| \begin{matrix} r_1(t)=t\vec{i}+0\cdot\vec{j} \\ |r_1'(t)|=1 \end{matrix} \right| = \int_0^1 h(t,0)\,dt$$

$$\int_{\Gamma_2} h\,ds = \left| \begin{matrix} r_2(t)=\vec{i}+t\vec{j} \\ r_2'(t)=1 \end{matrix} \right| = \int_0^1 h(1,t)\,dt$$

Weak formulation: Find $u=u(x,y)$ such that

$$\int_0^1\int_0^x \underbrace{\nabla u\cdot\nabla v}_{(1+x+y)\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]}\,dy\,dx \overbrace{x(1+x+y)} + \int_0^1 2u(t,0)v(t,0)\,dt + \int_0^1 u(1,t)v(1,t)\,dt = \int_0^1\int_0^x 1\,v\,dy\,dx + \int_0^1 10v(t,0)\,dt + \int_0^1 5v(1,t)\,dt$$

for all test function $v=v(x,y)$.

$\Sigma=0.5$

(c) ϕ_1: $\quad \phi_1(x,y)=ax+by+c$

$$\left.\begin{matrix}\phi_1(0,0)=1=c\\ \phi_1(1,0)=0=a+c\\ \phi_1(1,1)=0=a+b+c\end{matrix}\right\}\left.\begin{matrix}c=1\\ a=-1\\ b=0\end{matrix}\right\}\begin{matrix}\phi_1(x,y)=1-x\\ \nabla\phi_1=-\vec{i}\end{matrix}$$

0.2

$\Sigma=0.6$

ϕ_2: $\quad \phi_2(x,y)=ax+by+c$

$$\left.\begin{matrix}\phi_2(0,0)=0=c\\ \phi_2(1,0)=1=a+c\\ \phi_2(1,1)=0=a+b+c\end{matrix}\right\}\left.\begin{matrix}c=0\\ a=1\\ b=-1\end{matrix}\right\}\begin{matrix}\phi_2(x,y)=x-y\\ \nabla\phi_2=\vec{i}-\vec{j}\end{matrix}$$

0.2

ϕ_3: $\quad \phi_3(x,y)=ax+by+c$

$$\left.\begin{matrix}\phi_3(0,0)=0=c\\ \phi_3(1,0)=0=a+c\\ \phi_3(1,1)=1=a+b+c\end{matrix}\right\}\left.\begin{matrix}c=0\\ a=c\\ b=1\end{matrix}\right\}\begin{matrix}\phi_3(x,y)=y\\ \nabla\phi_3=\vec{j}.\end{matrix}$$

0.2

(d) $a_{ij} = \iint_\Omega \lambda \nabla\phi_i \cdot \nabla\phi_j \, dA + \int_\Gamma \kappa \phi_i \phi_j \, ds$ ③

$$a_{11} = \int_0^1 \int_0^x (1+x+y)\,1\,dy\,dx + 2\int_0^1 (1-t)(1-t)\,dt + \int_0^2 0\cdot 0\,dt =$$

$$= \int_0^1 \left[y + xy + \frac{y^2}{2}\right]_0^x dx + 2\left[\frac{t^3}{3}\right]_0^1 = \int_0^1 x + x^2 + \frac{x^2}{2}\,dx + \frac{2}{3} = \left[\frac{x^2}{2} + \frac{3}{2}\frac{x^3}{3}\right]_0^1 + \frac{2}{3} \quad 0.2$$

$$= 1 + \frac{2}{3} = \boxed{\frac{5}{3}}$$

$$a_{22} = \int_0^1 \int_0^x (1+x+y)\cdot 2\,dy\,dx + 2\int_0^1 t^2\,dt + \int_0^1 (1-t)^2\,dt = 2\cdot 1 + \frac{2}{3} + \frac{1}{3} = \boxed{3} \quad 0.2$$

$\Sigma = 1.2$

$$a_{33} = \int_0^1 \int_0^x (1+x+y)\cdot 1\,dy\,dx + 2\int_0^1 0\cdot 0\,dt + \int_0^1 t^2\,dt = 1 + \frac{1}{3} = \boxed{\frac{4}{3}} \quad 0.2$$

$$a_{12} = \int_0^1 \int_0^x (1+x+y)(-1)\,dy\,dx + 2\int_0^1 (1-t)\,t\,dt + \int_0^1 0\cdot(1-t)\,dt = -1 + 2\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = -1 + \frac{1}{3} = \boxed{-\frac{2}{3}} \quad 0.2$$

$$a_{13} = \int_0^1 \int_0^x (1+x+y)\cdot 0\,dy\,dx + 2\int_0^1 (1-t)\,0\,dt + \int_0^1 0\cdot t\,dt = \boxed{0} \quad 0.2$$

$$a_{23} = \int_0^1 \int_0^x (1+x+y)(-1)\,dy\,dx + 2\int_0^1 t\cdot 0\,dt + \int_0^1 (1-t)\,t\,dt = -1 + \frac{1}{6} = \boxed{-\frac{5}{6}} \quad 0.2$$

(f) $b_j = \iint_\Omega f\phi_j \, dA + \int_\Gamma (g + \kappa u_\infty)\phi_j \, ds$

$$b_1 = \int_0^1 \int_0^x 1 - x\,dy\,dx + \int_0^1 10(1-t)\,dt + \int_0^1 5\cdot 0\,dt = \int_0^1 [y - xy]_0^x\,dx + 10\left[\frac{t^2}{2}\right]_0^1 \quad 0.2$$

$$= \int_0^1 x - x^2\,dx + 5 = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + 5 = 5 + 1/6 = \frac{31}{6}$$

$$b_2 = \int_0^1 \int_0^x x - y\,dy\,dx + \int_0^1 10\,t\,dt + \int_0^1 5(1-t)\,dt = \int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx + 10\cdot\frac{1}{2} + 5\cdot\frac{1}{2}$$

$\Sigma = 0.6$

$$= \int_0^1 \underbrace{x^2 - \frac{x^2}{2}}_{\frac{x^2}{2}}\,dx + \frac{15}{2} = \frac{1}{6} + \frac{15}{2} = \frac{1}{6} + \frac{45}{6} = \frac{46}{6} = \frac{23}{3} \quad 0.2$$

$$b_3 = \int_0^1 \int_0^x y\,dy\,dx + \int_0^1 10\cdot 0\,dt + \int_0^1 5\cdot t\,dt = \int_0^1 \frac{x^2}{2}\,dx + \frac{5}{2} = \frac{1}{6} + \frac{5}{2} = \frac{1}{6} + \frac{15}{6} = \frac{16}{6} = \frac{8}{3}$$

0.2

(e) $m_{11} = \int_0^1\int_0^x (1-x)^2\,dy\,dx = \int_0^1 (1-x)^2\cdot x\,dx = \int_0^1 x - 2x^2 + x^3\,dx = \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_{x=0}^1$ 0.1

$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{6-8+3}{12} = \boxed{\frac{1}{12}}$ 0.2

$m_{22} = \int_0^1\int_0^x (x-y)^2\,dy\,dx = \int_0^1 -\left[\frac{(x-y)^3}{3}\right]_{y=0}^x dx = \int_0^1 \frac{x^3}{3}\,dx = \left[\frac{x^4}{12}\right]_0^1 = \boxed{\frac{1}{12}}$ 0.2

$m_{33} = \int_0^1\int_0^x y^2\,dy\,dx = \int_0^1\left[\frac{y^3}{3}\right]_0^x dx = \int_0^1 \frac{x^3}{3}\,dx = \boxed{\frac{1}{12}}$ 0.2 $\Sigma = 1.2$

$m_{12} = \int_0^1\int_0^x (1-x)(x-y)\,dy\,dx = \int_0^1\int_0^x x - x^2 + xy - y\,dy\,dx = \int_0^1\left[xy - x^2y + x\frac{y^2}{2} - \frac{y^2}{2}\right]_0^x dx$

$= \int_0^1 x^2 - x^3 + \frac{x^3}{2} - \frac{x^2}{2}\,dx = \int_0^1 \frac{x^2}{2} - \frac{x^3}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{1}{24}}$ 0.2

$m_{23} = \int_0^1\int_0^x (x-y)y\,dy\,dx = \int_0^1\int_0^x xy - y^2\,dy\,dx = \int_0^1\left[\frac{xy^2}{2} - \frac{y^3}{3}\right]_0^x dx = \int_0^1 \frac{x^3}{2} - \frac{x^3}{3}\,dx$

$= \int_0^1 \frac{x^3}{6}\,dx = \left.\frac{x^4}{24}\right]_{x=0}^1 = \boxed{\frac{1}{24}}$ 0.2

$m_{13} = \int_0^1\int_0^x (1-x)y\,dy\,dx = \int_0^1\int_0^x y - xy\,dy\,dx = \int_0^1\left[\frac{y^2}{2} - x\frac{y^2}{2}\right]_0^x dx = \int_0^1\left(\frac{x^2}{2} - \frac{x^3}{2}\right)dx = \boxed{\frac{1}{24}}$

0.2

(g)

$$\begin{bmatrix} \frac{5}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & 3 & -\frac{5}{6} \\ 0 & -\frac{5}{6} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{31}{6} \\ \frac{23}{3} \\ \frac{8}{3} \end{bmatrix}$$

null $\Sigma = 0.1$

$\Sigma = 5.0$