MVE515 Bonus set 1 solutions

(1.1) Since $u(-1,t)=1$ is a Dirichlet boundary condition we chose a test function v such that $v(-1)=0$. Multiplying the equation by v and integrating:

$$\int_{-1}^{1} t(1-x^2)\,v(x)\,dx = \int_{-1}^{1} D_t u(x,t)\,v(x)\,dx - \int_{-1}^{1} D_x\big((1+x^2)D_x u(x,t)\big)^2 \cdot v(x)\,dx$$

$$= \int_{-1}^{1} D_t u(x,t)\,v(x)\,dx - \Big(\Big[(1+x^2)D_x u(x,t)\cdot v(x)\Big]_{x=-1}^{1} - \int_{-1}^{1}(1+x^2)D_x u(x,t)\,D_x v(x)\,dx$$

$$= \int_{-1}^{1} D_t u(x,t)\cdot v(x)\,dx + \int_{-1}^{1}(1+x^2)D_x u(x,t)\,D_x v(x)\,dx - \Big(2\underbrace{D_x u(1,t)}_{5-2u(1,t)}v(1) - \underbrace{2D_x u(-1,t)v(-1)}_{=0}\Big)$$

$$= + \int_{-1}^{1} D_t u(x,t)\,v(x)\,dx + \int_{-1}^{1}(1+x^2)D_x u(x,t)\cdot D_x v(x)\,dx - 2(5-2u(1,t))\cdot v(1)$$

$$\Rightarrow \int_{-1}^{1} D_t u(x,t)\,v(x) + \int_{-1}^{1}(1+x^2)D_x u(x,t)\,D_x v(x)\,dx + 4u(1,t)\,v(1)$$
$$= \int_{-1}^{1} t(1-x^2)\,v(x)\,dx + 10\,v(1)$$

Weak formulation: Find $u=u(x,t)$ such that $u(x,0)\overset{0.1p}{=}e^x$, $\underset{0.1p}{u(-1,t)=1}$ and u satisfies

$$\int_{-1}^{1} D_t u(x,t)\cdot v(x)\,dx + \int_{-1}^{1}(1+x^2)D_x u(x,t)\,D_x v(x)\,dx + 4u(1,t)v(1) = \int_{-1}^{1} t(1-x^2)v(x)\,dx + 10v(1).$$

0.7p

for all v with $v(-1)=0$.
0.1p

$\Sigma = 1p$

(1.2)

$$\begin{cases} -D((2-x)Du) = x^2+x & \text{in } (0,1) \\ u'(0)=1 \qquad u'(1)+3u(1)=2 \end{cases}$$

$$-D(2-x)Du = x^2+x$$

$$(x-2)Du = \frac{x^3}{3}+\frac{x^2}{2}+c$$

$$Du = \frac{1}{3}\frac{x^3}{x-2}+\frac{1}{2}\frac{x^2}{x-2}+\frac{c}{x-2} \qquad 0.2$$

Long division or change of variables:

(a) $\int \frac{x^3}{x-2}\,dx = \left| \begin{matrix} t=x-2 \\ dt=dx \end{matrix} \right| = \int \frac{(t+2)^3}{t}\,dt = \int \frac{t^3+6t^2+12t+8}{t}\,dt$

$= \int t^2+6t+12+\frac{8}{t}\,dt = \frac{t^3}{3}+3t^2+12t+8\ln|t|+C_1$

$= \frac{(x-2)^3}{3}+3(x-2)^2+12(x-2)+8\ln|x-2|+C_2$

(b) $\int \frac{x^2}{x-2}\,dx = \left| \begin{matrix} t=x-2 \\ dt=dx \end{matrix} \right| = \int \frac{(t+2)^2}{t}\,dt = \int \frac{t^2+4t+4}{t}\,dt$

$= \int t+4+\frac{4}{t}\,dt = \frac{t^2}{2}+4t+4\ln|t|+C = \frac{(x-2)^2}{2}+4(x-2)+4\ln|x-2|+C_2$

$\Rightarrow$

$$u(x) = \frac{(x-2)^3}{9}+(x-2)^2+4(x-2)+\frac{8}{3}\ln|x-2|+\frac{(x-2)^2}{4}+2(x-2) \qquad 0.4$$
$$+\,2\ln|x-2|+c\ln|x-2|+D$$

$x\in(0,1)$

$\Downarrow$

$$\frac{(x-2)^3}{9}+\frac{5}{4}(x-2)^2+6(x-2)+\Big(\overbrace{\frac{14}{3}+c}^{8/3}\Big)\ln(2-x)+\underset{\frac{59}{12}}{\underset{\parallel}{D}}$$

Constants:

$1 = u'(0) = \frac{c}{-2} \Rightarrow \boxed{c=-2} \qquad 0.2$

$2 = -\frac{1}{3}-\frac{1}{2}+2+3\left(-\frac{1}{9}+\frac{3}{4}-6+D\right)$

$\Rightarrow D = \boxed{\frac{59}{12}} \qquad 0.2$

$\Sigma = 1{,}0$

(1.3)

$$V = \int_1^3\int_1^2 \frac{1}{1+x+y}\,dy\,dx = \int_1^3 \ln(1+x+y)\Big]_{y=1}^{y=2}\,dx$$

$$= \int_1^3 \ln(3+x)\,dx - \int_1^3 \ln(2+x)\,dx \qquad 0.2$$

Note: $\int 1\cdot\ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - x + C$

$$= \Big[(3+x)\ln(3+x) - (3+x)\Big]_{x=1}^{3} - \Big[(2+x)\ln(2+x) - (2+x)\Big]_{x=1}^{3} \qquad 0.6$$

$\Sigma = 1p$

$$= 6\ln 6 - 6 - 4\ln 4 + 4 - (5\ln 5 - 5 - 3\ln 3 + 3)$$

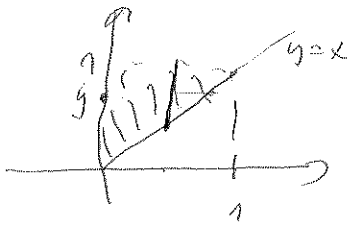
$$= 6\ln 6 - 4\ln 4 - 5\ln 5 + 3\ln 3 = \ln\frac{6^6\cdot 3^3}{4^4\cdot 5^5} \approx 0.454 \qquad 0.2$$

(1.4)

$$\int_0^1\int_x^1 e^{x/y}\,dy\,dx = \iint_D e^{x/y}\,dA$$

where $D = \{(x,y) : x\in[0,1],\ x\le y\le 1\}$

0.6 $\quad = \{(x,y) : y\in[0,1]\ \ 0\le x\le y\}$

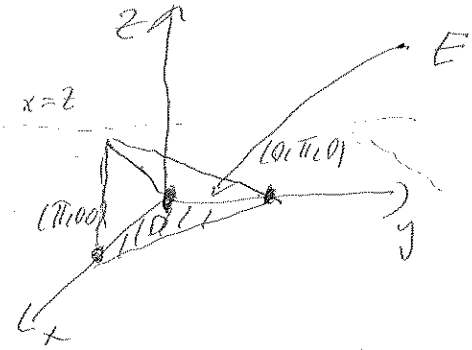


$$\Rightarrow \int_0^1\int_x^1 e^{x/y}\,dy\,dx = \int_0^1\int_0^y e^{x/y}\,dx\,dy = \int_0^1 y\,e^{x/y}\Big]_{x=0}^{x=y}\,dy$$

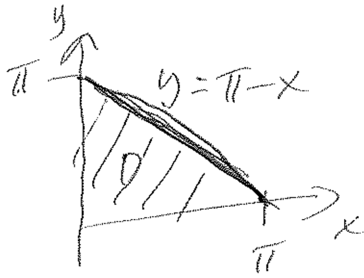
$$= \int_0^1 ye - y\,dy \ \ 0.2 = \Big[(e-1)\frac{y^2}{2}\Big]_{y=0}^{1} = \frac{e-1}{2} \quad 0.2$$

$\Sigma\ 1p$

1.5



$E = \{ (x,y,z) : (x,y) \in D \quad 0 \le z \le x \}$



$D = \{ (x,y) \mid 0 \le x \le \pi \quad 0 \le y \le \pi - x \}$

$$\Rightarrow \iiint_E \cos y \, dV = \int_0^{\pi} \int_0^{\pi - x} \int_0^{x} \cos y \, dz \, dy \, dx \quad 0.6$$

$$= \int_0^{\pi} \int_0^{\pi - x} x \cdot \cos y \, dy \, dx = \int_0^{\pi} x \left[\sin y \right]_{y=0}^{y=\pi-x} dx$$

$$= \int_0^{\pi} x \cdot \sin(\pi - x) \, dx = x \cos(\pi - x) \Big]_{x=0}^{\pi} - \int_0^{\pi} \cos(\pi - x) \, dx$$

0.2

$$= \pi + \sin(\pi - x) \Big]_{x=0}^{\pi} = \underline{\underline{\pi}} \quad 0.2$$

$\Sigma = 1p$