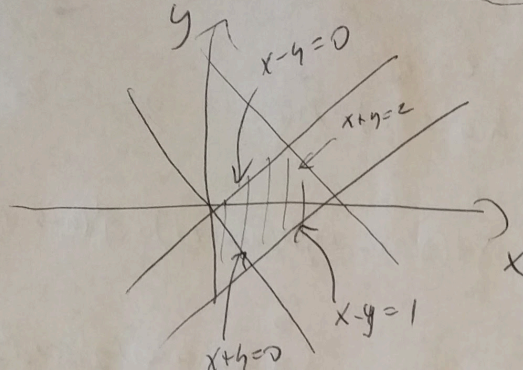
Bonus point set 2 solution

2.1



$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \quad x = \frac{u+v}{2}, \quad y = \frac{u-v}{2} \qquad 0.2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \qquad 0.2$$

$$\iint_D (x+y)e^{x^2-y^2}\,dA = \int_0^1\int_0^2 u e^{u\cdot v} \cdot \frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^2\int_0^1 u e^{uv}\,dv\,du \qquad 0.2$$

$\Sigma = 1p$

$$= \frac{1}{2}\int_0^2 \left[e^{uv}\right]_{v=0}^{1} du = \frac{1}{2}\int_0^2 e^u - 1\,du = \frac{1}{2}\left(e^u - u\right)_0^2 = \frac{1}{2}\left(e^2 - 2 - 1\right) = \frac{1}{2}(e^2-3)$$

0.2 0.2

2.2

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw \qquad 0.2$$

$$V = \iiint_D 1\,dV = \int_0^1\int_0^{1-u}\int_0^{1-u-v} 8uvw\,dw\,dv\,du \qquad 0.2$$

$$= 8\int_0^1\int_0^{1-u} u\cdot v(1-u-v)^2 \cdot \frac{1}{2}\,dv\,du = 4\int_0^1 u\int_0^{1-u} v(1+u^2+v^2-2u-2v+2uv)\,dv\,du \qquad 0.2$$

$$= 4\int_0^1 u\int_0^{1-u} v + u^2v + v^3 - 2uv - 2v^2 + 2uv^2\,dv\,du$$

$$= 4\int_0^1 u\left(\frac{(1-u)^2}{2} + u^2\frac{(1-u)^2}{2} - 2u\frac{(1-u)^2}{2} - 2\frac{(1-u)^3}{3} + 2u\frac{(1-u)^3}{3} + \frac{(1-u)^4}{4}\right)du$$

$\Sigma = 1p$

$= 0.01$ 0.4

Wolfram α.

2.3

(a) $C: \; x = g(t) \quad y = h(t) \qquad a \le t \le b$

$-C: \; x = g(a+b-t) = \tilde{g}(t) \qquad y = h(a+b-t) = \tilde{h}(t) \qquad a \le t \le b.$ 0.3

$$\int_{-C} f(x,y)\,dx = \int_a^b f(\tilde{g}(t), \tilde{h}(t)) \cdot \frac{d}{dt}\tilde{g}(t)\,dt = \int_a^b f(g(a+b-t), h(a+b-t)) \cdot g'(a+b-t) \cdot (-1)\,dt$$

$$= \left| \begin{matrix} a+b-t = s \\ -dt = ds \end{matrix} \right| = \int_b^a f(g(s), h(s))\, g'(s)\,ds = -\int_a^b f(g(s), h(s))\, g'(s)\,ds = -\int_C f(x,y)\,dx$$

(b) Analogous to (a). 0.3

(c)

$$\int_{-C} f(x,y)\,ds = \int_a^b f(\tilde{g}(t), \tilde{h}(t)) \cdot \sqrt{\left(\frac{d\tilde{g}}{dt}\right)^2 + \left(\frac{d\tilde{h}}{dt}\right)^2}\,dt = \int_a^b f(g(a+b-t), h(a+b-t))$$

$$\cdot \sqrt{g'(a+b-t)^2 + h'(a+b-t)^2}\,dt = \left| \begin{matrix} a+b-t = s \\ -dt = ds \end{matrix} \right|$$

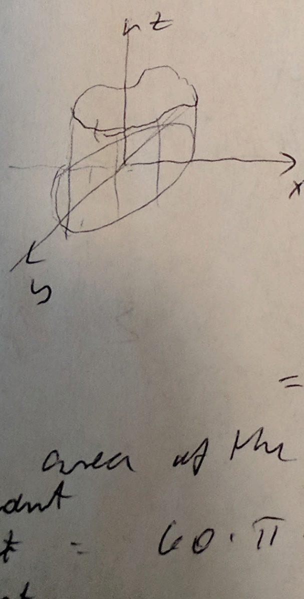
$\Sigma = 1p$

0.4

$$= \int_b^a f(g(s), h(s)) \sqrt{\left(\frac{dg}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2}\,(-1)\,ds$$

$$= \int_a^b f(g(s), h(s)) \sqrt{\left(\frac{dg}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2}\,ds = \int_C f(x,y)\,ds.$$

2.4



The area of the fence (1 side): 0.4

$$\int_C h(x,y)\,ds = \int_0^{2\pi} 3 + 0.04\,(25 \cdot \cos^2 t - 25\sin^2 t) \cdot \underbrace{\sqrt{25(\sin^2 t + \cos^2 t)}}_{5}\,dt$$

$$= \int_0^{2\pi} 15 + 5\,(\overbrace{\cos^2 t - \sin^2 t}^{\cos 2t})\,dt$$

$$= 2\pi \cdot 15 + 5 \cdot \frac{1}{2} \overbrace{\left[\sin 2t\right]_0^{2\pi}}^{0} = 30\pi.$$ 0.2

The area of the fence (both sides): 60π. 0.2

Total amount of paint $= 60 \cdot \pi \cdot \frac{1}{50} = \frac{6}{5} \cdot \pi$ 0.2

$\Sigma = 1p$

2.5 The force field is conservative. A potential of $\vec{F}$ is

$$f = \frac{-k}{\sqrt{x^2+y^2+z^2}}$$ 0.5

$$W = \int_L \vec{F}\, d\vec{r} = f(1,2,4) - f(0,2,0) = -k\left(\frac{1}{\sqrt{1+4+16}} - \frac{1}{\sqrt{4}}\right)$$

$$= \frac{k}{2} - \frac{k}{\sqrt{21}}.$$ 0.5

$\Sigma = 1p$