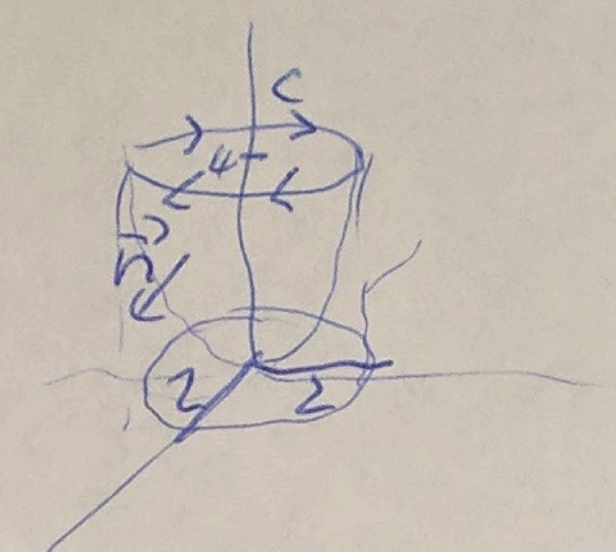
Bonus Problems 4 solutions. (1)

(4.1) $\vec{F} = y^2 \cdot z \cdot \vec{i} - 2y\,\vec{j} + x\,\vec{k}$

$z = g(x,y) = x^2 + y^2$



$\vec{r}(t) = 2\cos(2\pi - t)\,\vec{i} + 2\sin(2\pi - t)\,\vec{j} + 4\,\vec{k}$ $\qquad 0 \le t \le 2\pi$

$r'(t) = 2\sin(2\pi - t)\,\vec{i} - 2\cos(2\pi - t)\,\vec{j} + 0 \cdot \vec{k}$

(clockwise when viewed from above!)

0.2

$$\int_C \vec{F}\,d\vec{r} = \int_0^{2\pi} \vec{F}(r(t)) \cdot r'(t)\,dt =$$

$$= \int_0^{2\pi} \left(4\sin^2(2\pi - t) \cdot 4\,\vec{i} - 4\sin(2\pi - t)\,\vec{j} + 2\cos(2\pi - t) \cdot \vec{k}\right) \cdot \left(2\sin(2\pi - t)\,\vec{i} - 2\cos(2\pi - t) \cdot \vec{j} + 0\,\vec{k}\right) dt$$

$$= \int_0^{2\pi} 32\sin^3(2\pi - t) + 8\underbrace{\sin(2\pi - t) \cdot \cos(2\pi - t)}_{4\sin(4\pi - 2t)}\,dt$$

$$= 32\int_0^{2\pi} \sin^2(2\pi - t) \cdot \sin(2\pi - t)\,dt + \underbrace{4\int_0^{2\pi} \sin(4\pi - t)\,dt}_{0}$$

$$= 32\int_0^{2\pi} (1 - \cos^2(2\pi - t))\sin(2\pi - t)\,dt$$

$$= 32\underbrace{\int_0^{2\pi} \sin(2\pi - t)\,dt}_{0} - \int_0^{2\pi} \cos^2(2\pi - t) \cdot \sin(2\pi - t)\,dt$$

$$= -\left[\frac{32\cos^3(2\pi - t)}{3}\right]_0^{2\pi} = 0.$$ 0.5

Next

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & -2y & x \end{vmatrix} = \vec{i}(0 - 0) - \vec{j}(1 - y^2) - \vec{k}\,2yz$$

The normal (upward) of S:

$$\vec n = -\frac{\partial g}{\partial x}\vec i - \frac{\partial g}{\partial y}\vec j + \vec k = -2x\vec i + -2y\vec j + \vec k$$

The downward normal is 0.2

$$-\vec n = 2x\vec i + 2y\cdot\vec j - \vec k.$$ let $D = \{(x,y) : x^2+y^2 \le 4\}$

$$\iint_S \text{curl}\vec F \cdot d\vec S = \iint_D (y^2-1)\vec j - \vec k(y(x^2+y^2)) \cdot (2x\vec i + 2y\vec j - \vec k)\, dA$$

$$= \iint_D \cancel{2y^3} - 2y + 2yx^2 + \cancel{2y^3}\, dA$$

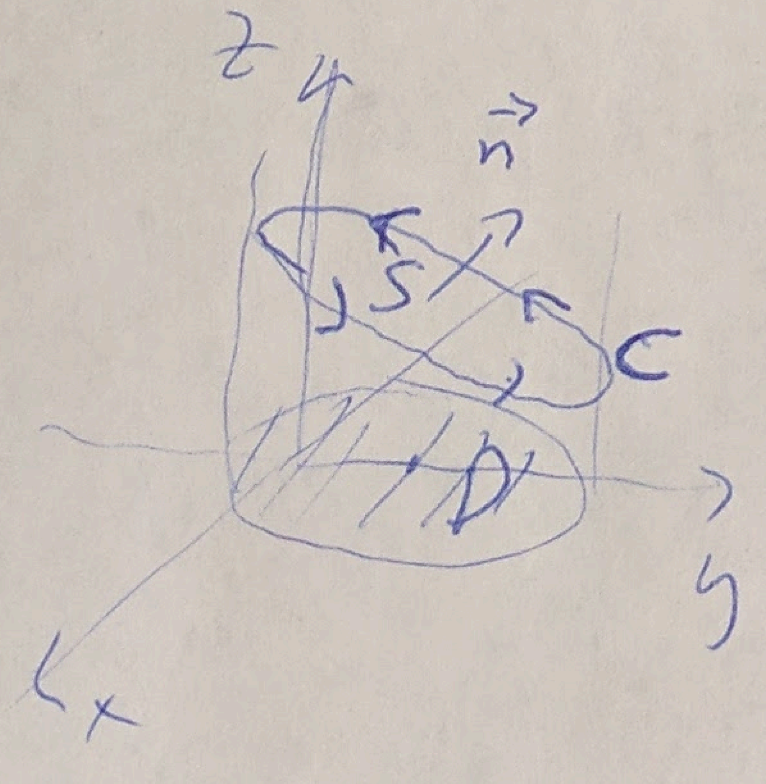
$$-\iint_D 2y + 2yx^2\, dA = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right| = -2\int_0^{2\pi}\int_0^2 (r\sin\theta + r^3\sin\theta\cdot\cos^2\theta)\, r\, dr\, d\theta \quad + 4r^3\sin^3\theta$$

$$= 4\underbrace{\int_0^{2\pi}\sin^3\theta\, d\theta}_{0}\cdot\int_0^2 r^4\, dr + -2\underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{0}\cdot\int_0^2 r^2\, dr - 2\underbrace{\int_0^{2\pi}\sin\theta\cdot\cos^2\theta\, d\theta}_{0}\int_0^2 r^4\, dr$$

Σ 1.4p

$= 0.$ 0.5

(4.2) $\vec F = (z^2 + y^2 + \sin(x^2))\vec i + (2xy+z)\vec j + (xz + 2yz)\cdot\vec k$



S: $(x,y) \in D$ $\quad 0 \le z \le 3 - y - 2x = g(x,y)$

$D = \{(x,y) \mid (x-1)^2 + 4y^2 \le 16\}$

Stokes: $\int_C \vec F\, d\vec r = \iint_S \text{curl}\,\vec F \cdot d\vec S$

$$\text{curl}\,\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2+y^2+\sin(x^2) & 2xy+z & xz+2yz \end{vmatrix} = (2z-1)\vec i + (z - \cancel{2z})\vec j + (2y - 2y)\vec k$$

0.3

$$= (2z-1)\vec i + z\cdot\vec j$$

upward normal:

$$\vec{n} = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} = 2\vec{i} + \vec{j} + \vec{k} \qquad 0.3$$

$$\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_D \left(2(3-y-2x)-1\right)\cdot 2 + (3-y-2x)\, dA$$

$$= \iint_D 13 - 5y - 10x\, dA = \left|\begin{array}{l} x-1 = r\cos\theta \Rightarrow x = 1 + r\cos\theta \\ 2y = r\sin\theta \Rightarrow y = \frac{r}{2}\sin\theta \\ 0 \le r \le 4 \quad 0 \le \theta \le 2\pi \\ \frac{\partial(x,y)}{\partial(r\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{1}{2}\sin\theta & \frac{r}{2}\cos\theta \end{vmatrix} = \frac{r}{2}\cos^2\theta + \frac{r}{2}\sin^2\theta = \frac{r}{2} \end{array}\right.$$

$$= \int_0^{2\pi}\int_0^4 \left(13 - \frac{5}{2}r\sin\theta - 10(1 + r\cos\theta)\right)\frac{r}{2}\, dr\, d\theta \qquad \Sigma = 1.1p$$

$$= \int_0^{2\pi}\int_0^4 \frac{3r}{2}\, dr\, d\theta + 0 = 2\pi\left[\frac{3r^2}{4}\right]_0^4 = 24\pi. \qquad 0.5$$

4.3 $\operatorname{div}\vec{F} = 2x + 2y + 2z$ $\quad 0.2 \quad$ $\vec{F} = x^2\vec{i} + y^2\cdot\vec{j} + z^2\cdot\vec{k}$

$$\iiint_{B_a} \operatorname{div} F\, dV = \left|\begin{array}{l} x = r\cdot\sin\phi\cdot\cos\theta \\ y = r\cdot\sin\phi\cdot\sin\theta \\ z = r\cdot\cos\phi \end{array}\right| = \int_0^{\pi}\int_0^{2\pi}\int_0^{a} \left(2r\sin\phi\cos\theta + 2r\sin\phi\cdot\sin\theta\right.$$

$$\left. + r\cos\phi\right)\cdot \underbrace{r^2\cdot\sin\phi}_{\frac{1}{2}\sin(2\phi)}\, dr\, d\theta\, d\varphi = \boxed{0} \qquad 0.5$$

S_a: $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi$. Outward normal:

$$\vec{n}(\phi,\theta) = \vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\phi,\theta) = a\sin^2\phi\cos\theta\,\vec{i} + a\sin^2\phi\cdot\sin\theta\,\vec{j} + a\sin\phi\cos\phi\cdot\vec{k}$$

$$\vec{F}(\vec{r})\cdot\vec{n} = a^2\sin^2\phi\cos^2\theta\cdot a\cdot\sin^2\phi\cos\theta + a^2\sin^2\phi\sin^2\theta\cdot a\sin^2\phi\cdot\sin\theta$$

$$+ a^2\cdot\cos^2\phi\, a\cdot\sin\phi\cos\phi \qquad 0.4$$

$$= a^3\sin^4\phi\cos^3\theta + a^3\sin^4\phi\sin^3\theta + a^3\sin\phi\cos^3\phi$$

(4)

$$\iint_{S_a} \vec{F}\, d\vec{S} = \int_0^{\pi}\int_0^{2\pi} a^3 \sin^4\phi \cos^3\theta + a^3 \sin^4\phi \cdot \sin^3\theta + a^3 \sin\phi \cos^3\phi \, d\theta d\phi$$

$$= a^3 \int_0^{\pi} \sin^4\phi\, d\phi \cdot \underbrace{\int_0^{2\pi} \cos^3\theta + \sin^2\theta\, d\theta}_{0} + a^3 \cdot 2\pi \int_0^{\pi} \sin\theta \cdot \cos^3\phi\, d\phi$$

$\Sigma = 1.5p$

$$= a^3 2\pi \cdot \left[-\frac{\cos^4\phi}{4}\right]_0^{\pi} = 0. \quad 0.4$$

4.4 Let

$$\vec{F}(x,y,z) = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz-cy & cx-az & ay-bx \end{vmatrix} = \vec{i}\, 2a + \vec{j}(2b) + \vec{k}\, 2c$$

$$= 2\vec{n} \quad 0.3$$

Stokes:

$$\oint_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$$

$$0.2$$

$$= \int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_S 2\vec{n} \cdot \vec{n}\, dS = 2\iint_S |\vec{n}|^2 dS = 2\iint_S 1\, dS = 2A(S) \quad 0.5$$

$$\Rightarrow \frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz = A(S).$$

$\Sigma = 1p$

$\boxed{\text{Total: } 5p}$