P.1 (a) As $u(1)=0$ we take a test function v with $v(1)=0$. We multiply the equation by v and integrate by parts:

$$\int_0^1 (4x+1)v(x)dx = -\int_0^1 D((2x+1)Du)v\,dx = -\left([(2x+1)Du\cdot v]_0^1 - \int_0^1 (2x+1)DuDv\,dx\right)$$

Now:

$$[(2x+1)Du\,v]_0^1 = 3Du(1)v(1) - Du(0)v(0) = -Du(0)v(0)$$
$$= (2u(0)+1)v(0)$$

$$\Rightarrow \int_0^1 (4x+1)v(x)dx = 2u(0)v(0) - v(0) + \int_0^1 (2x+1)DuDv\,dx$$

Weak formulation: Find $u=u(x)$ such that $u(1)=0$ and

$$\int_0^1 (2x+1)DuDv\,dx + 2u(0)v(0) = \int_0^1 (4x+1)v(x)dx + v(0)$$

for all test functions v with $v(1)=0$.

(b)
$$-(2x+1)Du = 2x^2+x+C = x(2x+1)+C$$

$$Du = -x - \frac{C}{2x+1}$$

$$u(x) = -\frac{x^2}{2} - C\ln(2x+1) + D$$

$$0 = u(1) = -\frac{1}{2} - C\ln(3) + D \Rightarrow -\ln(3)C + D = \frac{1}{2} \Rightarrow +\ln 9\,C - 2D = -1$$

$$1 = -u'(0) + 2u(0) = C + 2D \quad \Rightarrow \quad \underbrace{C + 2D = 1}_{C=0 \quad D=\frac{1}{2}}$$

$$\Rightarrow \boxed{u(x) = -\frac{x^2}{2} + \frac{1}{2}}$$

(2) To show: $\text{curl}\,\vec{F} = 0$ or not.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \vec{i}\left(6xyz^2 - 6xyz^2\right) - \vec{j}\left(3y^2z^2 - 3y^2z^2\right) + \vec{k}\left(2yz^3 - 2yz^3\right) = \vec{0}$$

$\Rightarrow$ $\vec{F}$ is conservative. There is f such that $\nabla f = \vec{F}$.

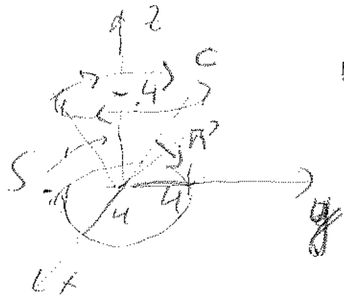
$$\frac{\partial f}{\partial x} = y^2z^3 \Rightarrow f = xy^2z^3 + g(y,z)$$

$$\frac{\partial f}{\partial y} = 2xyz^3 \qquad \frac{\partial f}{\partial y} = 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3 \Rightarrow g(y,z) = K(z)$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 \qquad \frac{\partial f}{\partial z} = 3xy^2z^2 + K'(z) = 3xy^2z^2 \Rightarrow K'(z) = 0 \Rightarrow K = C$$

$$\Rightarrow \boxed{f(x,y,z) = xy^2z^3 + C}$$

(3) To show: $\int_C \vec{F}\,d\vec{r} = \iint_S \text{curl}\,\vec{F}\,d\vec{S}$



when $z = 4 \Rightarrow x^2 + y^2 = 16$

$$\vec{F} = -y\vec{i} + x\vec{j} - 2\vec{k}$$

(a) C: $\vec{r}(t) = 4\cos(2\pi - t)\,\vec{i} + 4\sin(2\pi - t)\,\vec{j} + 4\vec{k}$

$\vec{r}'(t) = 4\sin(2\pi - t)\,\vec{i} - 4\cos(2\pi - t)\,\vec{j}$

(Negative orientation when viewed from above $\equiv$ positive orientation w.r.t. $\vec{n}$!)

$$\int_C \vec{F}\,d\vec{r} = \int_0^{2\pi} \left(-4\sin(2\pi - t)\vec{i} + 4\cos(2\pi - t)\vec{j} - 2\vec{k}\right) \cdot \left(4\sin(2\pi - t)\vec{i} - 4\cos(2\pi - t)\vec{j}\right) dt$$

$$= \int_0^{2\pi} \left(-16\sin^2(2\pi - t) - 16\cos^2(2\pi - t)\right) dt = -16 \int_0^{2\pi} 1\,dt = -32\pi.$$

(b) $S: \quad x = x \ , \ y = y \ , \ z = \sqrt{x^2+y^2} = g(x,y) \ ; \ (x,y) \in D = \{(x,y) : x^2+y^2 \le 16\}$

Downward normal:

$$\vec{n} = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} - \vec{k} = x(x^2+y^2)^{-1/2}\vec{i} + y(x^2+y^2)^{-1/2}\vec{j} - \vec{k}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -z \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}\cdot 2 = 2\vec{k}$$

$$\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_D 2\vec{k}\cdot\left(x(x^2+y^2)^{-1/2}\vec{i} + y(x^2+y^2)^{-1/2}\vec{j} - \vec{k}\right) dA = \iint_D -2\, dA$$

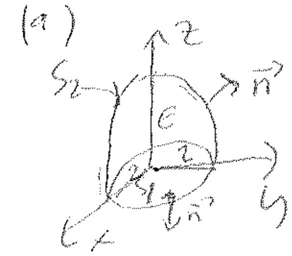
$$= -2\, A(D) = -2\cdot 4^2\pi = -32\pi .$$

Thus, $\displaystyle\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \int_C \vec{F}\, d\vec{r}$

(4) To show: $\displaystyle\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F}\, d\vec{S} \quad , \ \vec{F} = x^2\vec{i} + xy\vec{j} + z\vec{k}$

$z = 4 - x^2 - y^2$

(a)



$$\operatorname{div}\vec{F} = 2x + \cancel{x} + 1 = 3x + 1$$

$$\iint_{S_1}\int_0^{4-x^2-y^2} (3x+1)\, dz\, dA = \iint_{S_1} (3x+1)(4-x^2-y^2)\, dA$$

$$= \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right| = \int_0^{2\pi}\int_0^2 (3r\cos\theta + 1)(4-r^2)\, r\, dr\, d\theta$$

$$= 0 + \int_0^{2\pi}\int_0^2 4r - r^3\, dr\, d\theta = 2\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = 2\pi\left(8 - \frac{16}{4}\right) = 8\pi$$

(b) $S_1: \quad x = x \ , \ y = y \quad z = 0 = g$ Positive orientation = downward normal

$*\quad \vec{n} = +\frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} - \vec{k} = -\vec{k}$

$$\iint_{S_1} \vec{F}\, d\vec{S} = \iint_{S_1} 0\, dA = 0 .$$

S_2: Positive orientation = upward normal:

$x = x \quad y = y \quad z = 4 - x^2 - y^2 \qquad (x,y) \in D = \{(x,y) \mid x^2+y^2 \leq 4\}$

$\vec{n} = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} = 2x\vec{i} + 2y\vec{j} + \vec{k}$

$$\iint_{S_2} \vec{F}\, d\vec{S} = \iint_D (x^2\vec{i} + xy\vec{j} + (4-x^2-y^2)\vec{k})(2x\vec{i} + 2y\vec{j} + \vec{k})\, dA$$

$$= \iint_D 2x^3 + 2xy^2 + 4 - x^2 - y^2\, dA = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{array} \right|$$

$$= \int_0^{2\pi}\int_0^2 (2r^3\cos^3\theta + 2r^2\cos\theta\cdot\sin^2\theta + 4 - r^2)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \cos^3\theta\, d\theta \int_0^2 r^4\, dr + 2\int_0^{2\pi} \cos\theta\cdot\sin^2\theta\, d\theta \cdot \int_0^2 r^3\, dr + 2\pi\int_0^2 4r - r^3\, dr$$

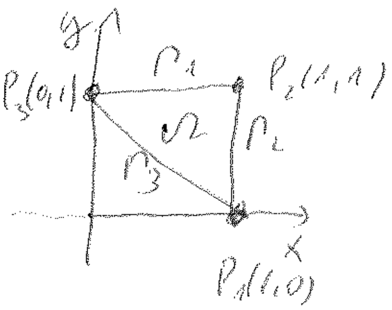
$=0$ Now ① $\int_0^{2\pi} \cos^3\theta\, d\theta = \int_0^{2\pi} (1-\sin^2\theta)\cos\theta\, d\theta = \int_0 \cos\theta\, d\theta$

$- \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta = 0 - \left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} = 0.$

② $\int_0^{2\pi} \cos\theta\sin^2\theta\, d\theta = \left(\frac{\sin^3\theta}{3}\right]_0^{2\pi} = 0.$

$$\Rightarrow \iint_{S_2} \vec{F}\, d\vec{S} = 2\pi\int_0^2 4r - r^3\, dr = 8\pi$$

5 (a)



$\lambda(x,y) = 2 + x + y$

$f(x,y) = 1 + xy$

$u_n = 2$

$B(x,y) = y + x - 1$

$g = 0.$

The Equation:

$$-\nabla\cdot(\lambda\nabla u) = -\nabla\cdot\left((2+x+y)\frac{\partial u}{\partial x}\vec{i} + (2+x+y)\frac{\partial u}{\partial y}\vec{j}\right) = -\frac{\partial}{\partial x}\left((2+x+y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left((2+x+y)\frac{\partial u}{\partial y}\right)$$

$$= -(2+x+y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 1+xy$$

Boundary condition:

Γ_1 : $y=1$ $0<x<1$, unit outward normal: $\vec{j}$

$$D_N u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\cdot\vec{j} = \frac{\partial u}{\partial y}$$

$$\lambda D_N u + \kappa(u-u_R) = (3+x)\frac{\partial u}{\partial y}(x,1) + x(u(x,1)-2) = 0 \qquad 0<x<1$$

Γ_2 : $x=1$ $0<y<1$. unit outward normal: $\vec{i}$

$$D_N u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\cdot\vec{i} = \frac{\partial u}{\partial x}$$

$$\lambda D_N u + \kappa(u-u_R) = (3+y)\frac{\partial u}{\partial x}(1,y) + y(u(1,y)-2) = 0$$

Γ_3 : $0<x<1$ ~~segment~~. $y = 1-x$, unit outward normal: $\frac{-1}{\sqrt{2}}(\vec{i}+\vec{j})$

$$D_N u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\left(-\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\right) = -\frac{1}{\sqrt{2}}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right).$$

$$\lambda D_N u + \kappa(u-u_R) = \frac{-3}{\sqrt{2}}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right) + 0 = \frac{-3}{\sqrt{2}}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 0.$$

Boundary value problem:

$$\begin{cases} \text{Find } u = u(x,y) \text{ such that} \\ -(2+x+y)\Delta u - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 1+xy & (x,y)\in\Omega \\ (3+x)\frac{\partial u}{\partial y}(x,1) + x(u(x,1)-2) = 0 & 0<x<1 \\ (3+y)\frac{\partial u}{\partial x}(1,y) + y(u(1,y)-2) = 0 & 0<y<1 \\ \frac{\partial u}{\partial x}(x,1-x) + \frac{\partial u}{\partial y}(x,1-x) = 0 & 0<x<1. \end{cases}$$

(b) Weak formulation: General weak formulation..

Find $u = u(x,y)$ such that

$$\iint_\Omega \lambda \nabla u \cdot \nabla v \, dA + \int_\Gamma \kappa u v \, ds = \iint_\Omega f v \, dA + \int_\Gamma (g + \kappa u_R)\, ds$$

Integration over Γ:

Γ_1: $\vec{r}(t) = t\vec{i} + \vec{j}$ $\quad \vec{r}'(t) = \vec{i}$ $\quad |\vec{r}'(t)| = 1$

$$\int_{\Gamma_1} h\,ds = \int_0^1 h(t,1)\,dt$$

Γ_2: $\vec{r}(t) = \vec{i} + t\vec{j}$ $\quad \vec{r}'(t) = \vec{j}$ $\quad |\vec{r}'(t)| = 1$

$$\int_{\Gamma_2} h\,ds = \int_0^1 h(1,t)\,dt$$

Γ_3: $h = 0$ on Γ_3!

Ω: $0<x<1$, $1-x<y<1$

Weak formulation: Find $u = u(x,y)$ such that

$$\left\{\begin{aligned} &\int_0^1\int_{1-x}^1 (2+x+y)\nabla u\cdot\nabla v\,dy\,dx + \int_0^1 t\,u(t,1)v(t,1)\,dt + \int_0^1 t\,u(1,t)v(1,t)\,dt \\ &\qquad = \int_0^1\int_{1-x}^1 (1+xy)v(x,y)\,dy\,dx + 2\int_0^1 t\,v(t,1)\,dt + 2\int_0^1 t\,v(1,t)\,dt \\ &\text{for all test functions } v. \end{aligned}\right.$$

(c) ϕ_1: $\phi_1(x,y) = ax + by + c$

$$\left.\begin{aligned} \phi_1(1,0) &= 1 = a + c \\ \phi_1(1,1) &= a+b+c = 0 \\ \phi_1(0,1) &= b + c = 0 \end{aligned}\right\}\quad \left.\begin{aligned} c = 1 &\Rightarrow b = -1 \\ &\Rightarrow a = 0 \end{aligned}\right\}\quad \begin{aligned} \phi_1(x,y) &= -y + 1 \\ \nabla\phi_1 &= -\vec{j} \end{aligned}$$

ϕ_2: $\phi_2(x,y) = ax + by + c$

$$\left.\begin{aligned} \phi_2(1,1) &= a+b+c = 1 \\ \phi_2(1,0) &= a + c = 0 \\ \phi_2(0,1) &= b + c = 0 \end{aligned}\right\}\quad \left.\begin{aligned} b &= 1 \\ c &= -1 \\ a &= 1 \end{aligned}\right\}\quad \begin{aligned} \phi_2(x,y) &= x + y - 1 \\ \nabla\phi_2 &= \vec{i} + \vec{j} \end{aligned}$$

ϕ_3: $\phi_3(x,y) = ax + by + c$

$$\left.\begin{aligned} \phi_3(0,1) &= 1 = b + c \\ \phi_3(1,1) &= a+b+c = 0 \\ \phi_3(1,0) &= a + c = 0 \end{aligned}\right\}\quad \left.\begin{aligned} a &= -1 \\ c &= 1 \\ b &= 0 \end{aligned}\right\}\quad \begin{aligned} \phi_3(x,y) &= 1 - x \\ \nabla\phi_3 &= -\vec{i} \end{aligned}$$

(d) $a_{13} = \int_0^1\int_{1-x}^1 (2+x+y)(-\vec{j})\cdot(-\vec{i})\,dy\,dx + \int_0^1 t\cdot 0\,(1-t)\,dt + \int_0^1 t(1-1)\cdot 0\,dt = 0.$