MVE515 Computational Mathematics-Practice exam

Problem 1.1.

(a) Write down the weak formulation of the following boundary value problem:

$$\begin{cases} -\mathrm{D}((x+1)\,\mathrm{D}u) = 2x+1 \quad x \in (0,1) \\ u(0) = 1, \quad u'(1) + 3u(1) = 2. \end{cases}$$

(b) Solve the boundary value problem from (a) by integrating twice.

Problem 1.2. Determine whether the vector field

$$\mathbf{F}(x,y,z) = \mathbf{i} + \sin z\,\mathbf{j} + y\cos z\,\mathbf{k}$$

is conservative. If yes, then find the general form of the potential f of $\mathbf{F}$.

Problem 1.3. Verify that Stoke's Theorem is true for the vector field

$$\mathbf{F}(x,y,z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k},$$

where S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward.

Problem 1.4. Verify that the Divergence Theorem is true for the vector field

$$\mathbf{F}(x,y,z) = \langle x^2, -y, z \rangle$$

and the solid cylinder E given by $y^2 + z^2 \leq 9$ and $0 \leq x \leq 2$.

Problem 1.5.

(a) Write down the boundary value problem for the 2D stationary heat equation on the triangle vertices with $P_1 = (0,0)$, $P_2 = (0,1)$, $P_3 = (1,1)$, with heat conductivity equals $3 - x - y$, constant heat source density equals 2, and constant ambient temperature equals 4 on all sides. The heat transfer coefficient equals $y - x$ at the boundary. There is no prescribed heat influx at the boundary.

(b) Write down the weak formulation of the problem.

(c) Write down the finite element basis functions ϕ_1, ϕ_2, ϕ_3 for a triangulation that consists of a single triangle $T = \Omega$ with nodes $P_1 = (0,0)$, $P_2 = (0,1)$, $P_3 = (1,1)$.

(d) Compute the elements of the stiffness matrix

$$a_{ij} = a_{ji} = \iint_{\Omega} \lambda \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \int_{\Gamma} \kappa\phi_i\phi_j \,\mathrm{d}s.$$

(e) Compute the elements of the load vector

$$b_j = \iint\limits_{\Omega} f\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} (g + \kappa u_A)\phi_j \,\mathrm{d}s.$$

(f) If the vector

$$\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

contains the nodal values of the finite element solution of the boundary value problem, write down the linear system of equations that needs to be solved to determine $\mathcal{U}$.

Problem 1.6. Write down the weak formulation of the following initial-boundary value problem in 3D.

$$\begin{cases} \partial_t u(x,y,z,t) - \Delta u(x,y,z,t) = 0, \quad (x,y,z) \in \Omega, \quad t > 0, \\ \qquad u(x,y,z,t) = u_{\mathrm{A}}, \quad (x,y,z) \in \Gamma_1, \quad t > 0, \\ \mathrm{D}_N u(x,y,z,t) + u(x,y,z,t) = g(x,y,z,t), \quad (x,y,z) \in \Gamma_2, \quad t > 0, \\ \quad u(x,y,z,0) = u_0(x,y,z), \end{cases}$$

where $\Delta u = \nabla \cdot \nabla u$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ (Γ_1 and Γ_2 are non-overlapping).