MVE 515 Re-exam solutions 2017 dec.

(1) $$-(\cos x\, u'(x))' = 1+x^2 \qquad x\in(0,1)$$

$$-u'(0)=1 \qquad \cos(1)\,u'(1)+u(1)=1$$

Take a test function $v=v(x)$ multiply the equation and integrate:

$$\int_0^1 (1+x^2)v(x)\,dx = -\int_0^1 (\cos x\, u'(x))'\,v(x)\,dx = -\left[\cos x\, u'(x) v(x)\right]_0^1 + \int_0^1 \cos x\, u' v'\,dx$$

Now $\left[\cos x\, u'(x) v(x)\right]_0^1 = \underbrace{\cos 1\, u'(1)}\, v(1) - \underbrace{u'(0)}\, v(0)$

$$= (1-u(1))v(1) + v(0)$$

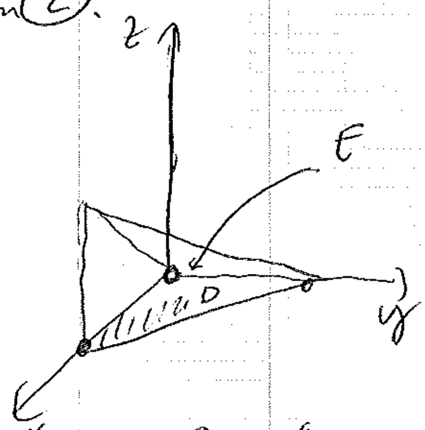
$$\Rightarrow \int_0^1 (1+x^2)v(x)\,dx = +(u(1)-1)v(1) - v(0) + \int_0^1 \cos x\, u'(x) v'(x)\,dx$$

<u>Weak formulation:</u> Find $u=u(x)$ such that

$$\int_0^1 \cos x\, u'(x) v'(x)\,dx + u(1)v(1) = \int_0^1 (1+x^2)v(x)\,dx + v(0)+v(1).$$

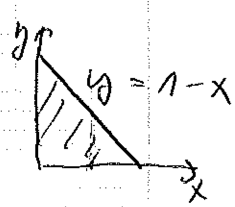
for all test functions v.

Problem (2).



$E=\{(x,y,z) : (x,y)\in D,\ 0\le z\le x\}$

D:



$D=\{0\le x\le 1,\ 0\le y\le 1-x\}$

$$\Rightarrow \iiint_V e^y\,dV = \int_0^1\int_0^{1-x}\int_0^x e^y\,dz\,dy\,dx$$

(2)

$$= \int_0^1 \int_0^{1-x} x e^y \, dy \, dx = \int_0^1 x e^y \Big]_{y=0}^{1-x} dx = \int_0^1 x e^{1-x} - x \, dx$$

$$= \int_0^1 x e^{1-x} dx - \int_0^1 x \, dx = \Big[x e^{1-x} \Big]_{x=0}^{1} + \int_0^1 e^{1-x} dx - \Big[\frac{x^2}{2} \Big]_0^1$$

$$= -1 \cdot e^0 + 0 + \Big[-e^{+1-x} \Big]_0^1 - \frac{1}{2} = -1 + (-1+e) - \frac{1}{2}$$

$$= -\frac{5}{2} + e$$

(3) Need to find $f = f(x,y,z)$ such that $\nabla f = F$

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f = -(x^2+y^2+z^2)^{-1/2} + C(y,z)$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}} \qquad \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial C}{\partial y}$$

$$\Downarrow$$

$$\frac{\partial f}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}} \qquad \frac{\partial C}{\partial y} = 0 \Rightarrow C = K(z)$$

$$\Downarrow$$

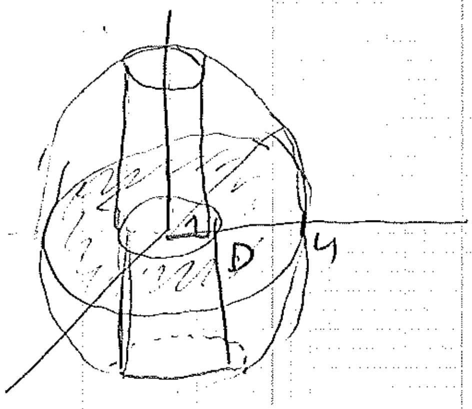
$$f = -(x^2+y^2+z^2)^{-1/2} + K(z)$$

$$\frac{\partial f}{\partial z} = + \frac{z}{(x^2+y^2+z^2)^{3/2}} + K'(z)$$

$$\Rightarrow K'(z) = 0 \Rightarrow K(z) = L$$

$$\Rightarrow f(x,y,z) = -(x^2+y^2+z^2)^{-1/2} + L = -\frac{1}{|r|} + L.$$

4



$D : \{(r,\theta) : 1 < r < 4 \;, 0 \le \theta \le 2\pi\}$

$$z = g(x,y) = \pm\sqrt{4-x^2-y^2} = \pm(4-x^2-y^2)^{1/2}$$

$$|r_x \times r_y| = \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}$$

$$= \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}.$$

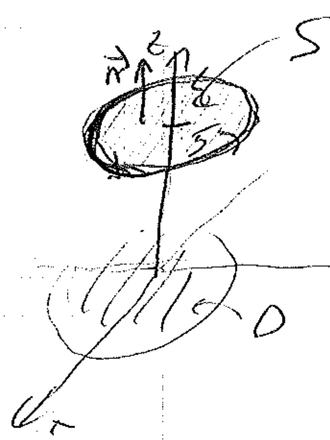
$$S = 2\int_0^{2\pi}\int_{+1}^{4}\sqrt{1+\frac{r^2\cos^2\theta + r^2\sin^2\theta}{4-r^2}}\; r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_1^4 \sqrt{\frac{4}{4-r^2}}\; r\,dr\,d\theta = 4\pi\cdot 2\int_1^4 (4-r^2)^{-1/2}\, r\,dr$$

$$= 8\pi\left(-\left[(4-r^2)^{1/2}\right]_1^4\right) = 8\pi\cdot\sqrt{3}.$$

5

$$\vec{F} = yz\,\vec{i} + 2xz\,\vec{j} + e^{xy}\,\vec{k}$$



$\vec{n} = \vec{k}$

$$\operatorname{curl} F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix}$$

$$= \vec{i}\,(xe^{xy}-2x) + -\vec{j}\,(ye^{xy}-y) + \vec{k}\,(2z-z) = (xe^{xy}-2x)\,\vec{i} + (y - e^{xy})\,\vec{j} + z\,\vec{k}.$$

$$\int_C F\cdot dr = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^4 5\cdot\vec{k}\cdot\vec{k}\; r\,dr\,d\varphi$$

$$= 10\pi\int_0^4 r\,dr = 10\pi\cdot\frac{r^2}{2}\Big]_0^4 = 80\pi.$$

6

$$\operatorname{div}(f\vec{F}) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

$$= f\frac{\partial P}{\partial x} + P\frac{\partial f}{\partial x} + f\frac{\partial Q}{\partial y} + Q\frac{\partial f}{\partial y} + f\frac{\partial R}{\partial z} + R\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + (P,Q,R)\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= f \operatorname{div}\vec{F} + \vec{F}\cdot\nabla f .$$

7 Take a test function v such that $v=0$ on Γ_1.

Then

$$\iiint_\Omega f v \, dV = -\iiint_\Omega \nabla\cdot(a\nabla u)\, v \, dV + \iiint_\Omega c u v \, dV = -\iint_\Gamma a D_N u \, v \, dS$$

$$+ \iiint_\Omega a\nabla u\cdot\nabla v \, dV + \iiint_\Omega c u v \, dV$$

$$= \cancel{-}\iint_{\Gamma_2} g v \, dS + \iiint_\Omega a\nabla u\cdot\nabla v \, dV + \iiint_\Omega c u v \, dV$$

Weak formulation: Find $u = u(x,y,z)$ such that

$u = u_A$ on Γ_1 and

$$\iiint_\Omega a\nabla u\cdot\nabla v \, dV + \iiint_\Omega c u v \, dV = \iiint_\Omega f v \, dV + \iint_{\Gamma_2} g v \, dS$$

~~whose~~ for every test function v with $v=0$ on Γ_1.

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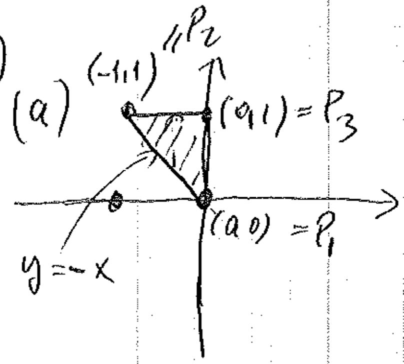
(a)



1. $$\phi_1(x,y) = ax + by + c$$

$$\left.\begin{aligned} \phi_1(0,0) &= c = 1 \\ \phi_1(-1,1) &= -a + b + c = 0 \\ \phi_1(0,1) &= b + c = 0 \end{aligned}\right\} \begin{aligned} c &= 1 \\ b &= -1 \\ a &= 0 \end{aligned} \Rightarrow \phi_1(x,y) = 1 - y$$

2. $$\phi_2(x,y) = ax + by + c$$

$$\left.\begin{aligned} \phi_2(-1,1) &= -a + b + c = 1 \\ \phi_2(0,0) &= c = 0 \\ \phi_2(0,1) &= b + c = 0 \end{aligned}\right\} \begin{aligned} a &= -1 \\ b &= 0 \\ c &= 0 \end{aligned} \Rightarrow \phi_2(x,y) = -x$$

3. $$\phi_3(x,y) = ax + by + c$$

$$\left.\begin{aligned} \phi_3(0,1) &= b + c = 1 \\ \phi_3(-1,1) &= -a + b + c = 0 \\ \phi_3(0,0) &= c = 0 \end{aligned}\right\} \begin{aligned} b &= 1 \\ a &= 1 \\ c &= 0 \end{aligned} \Rightarrow \phi_3(x,y) = x + y$$

(b)

$$\iint_\Omega \phi_1 \phi_2 \, dx = \int_{-1}^{0} \int_{-x}^{1} (1-y)(-x) \, dy \, dx = \int_{-1}^{0} \int_{x}^{1} (-x + xy) \, dy \, dx$$

$$= \int_{-1}^{0} \left[-xy + x\frac{y^2}{2} \right]_{y=-x}^{1} dx = \int_{-1}^{0} -x + \frac{x}{2} - \left(x^2 + \frac{x^3}{3}\right) dx = \int_{-1}^{0} -\frac{x^3}{3} - x^2 - \frac{x}{2} \, dx$$

$$= \left[-\frac{x^4}{12} - \frac{x^3}{3} - \frac{x^2}{4} \right]_{-1}^{0} = 0 - \left(-\frac{1}{12} + \frac{1}{3} + \frac{1}{4} \right) = \frac{1}{12} - \frac{4+3}{12} = -\frac{6}{12} = -\frac{1}{2}$$