MVE515 Exam sol. (2018 aug)

1 $\begin{cases} -D((2-x)Du) = x^2 & \text{in } (1, \tfrac{3}{2}) \\ u(1) = 1, \quad Du'(\tfrac{3}{2}) = 0 \end{cases}$

Sol. $(x-2)Du = \frac{x^3}{3} + C \Rightarrow Du = \frac{x^3}{3(x-2)} + \frac{C}{x-2}$ (1)

$\Rightarrow u = \int \frac{x^3}{3(x-2)}dx + C\ln|x-2| + D = \int \frac{(t+2)^3}{3t}dt + C\ln|x-2|$

$= \int \frac{t^3+6t^2+12t+8}{3t}dt + C\ln(2-x) = \int \frac{t^2}{3} + 2t + 4 + \frac{8}{3}\cdot\frac{1}{t}\,dt$

$+ C\ln(2-x) = \frac{t^3}{9} + t^2 + 4t + \frac{8}{3}\ln|t| + D + C\ln(2-x)$

$= \frac{(x-2)^3}{9} + (x-2)^2 + 4(x-2) + \left(\frac{8}{3} + C\right)\ln(2-x) + D$ (2)

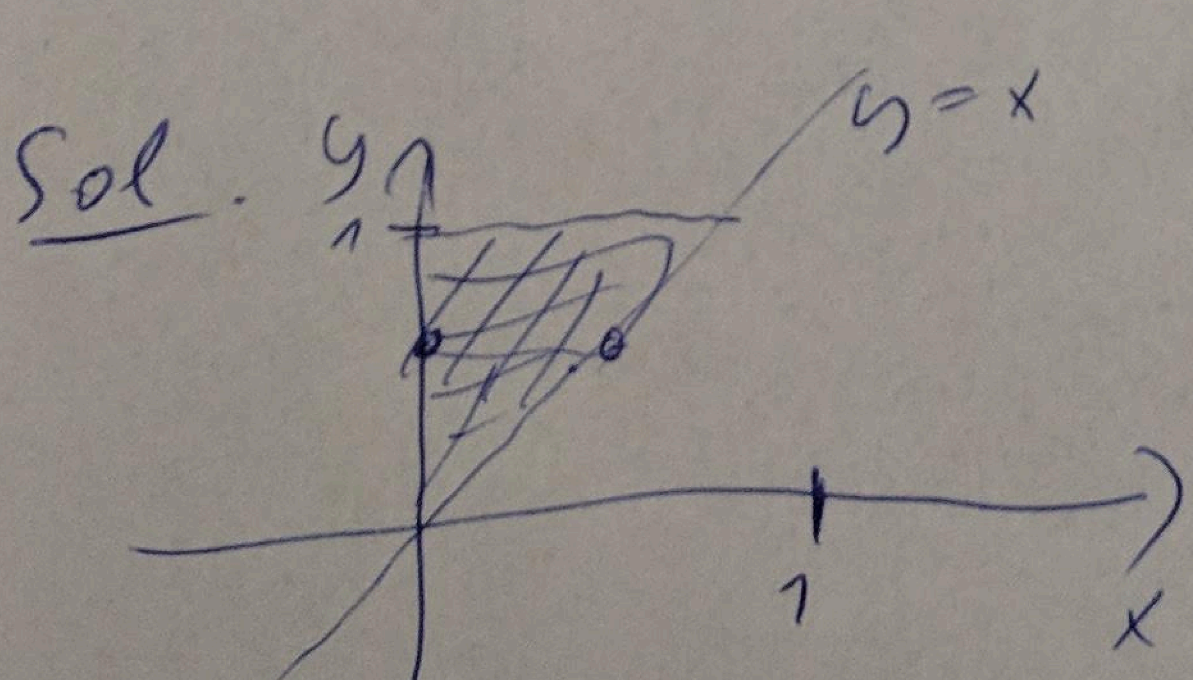
$u(1) = -\frac{1}{9} + 1 - 4 + D = 1 \Rightarrow D = \boxed{\frac{37}{9}}$ (2)

$u'(\tfrac{3}{2}) = \frac{(\frac{3}{2})^3}{3(\frac{3}{2}-2)} + \frac{C}{3/2-2} = -\left(\frac{3}{2}\right)^2 - 2C = 0 \Rightarrow C = \boxed{-\frac{9}{8}}$ (2)

$\Rightarrow u(x) = \frac{(x-2)^3}{9} + (x-2)^2 + 4(x-2) + \frac{37}{24}\ln(2-x) + \frac{37}{9}.$

2 $\int_0^1 \int_x^1 \cos\left(\frac{x}{y}\right) dy\,dx$

Sol.



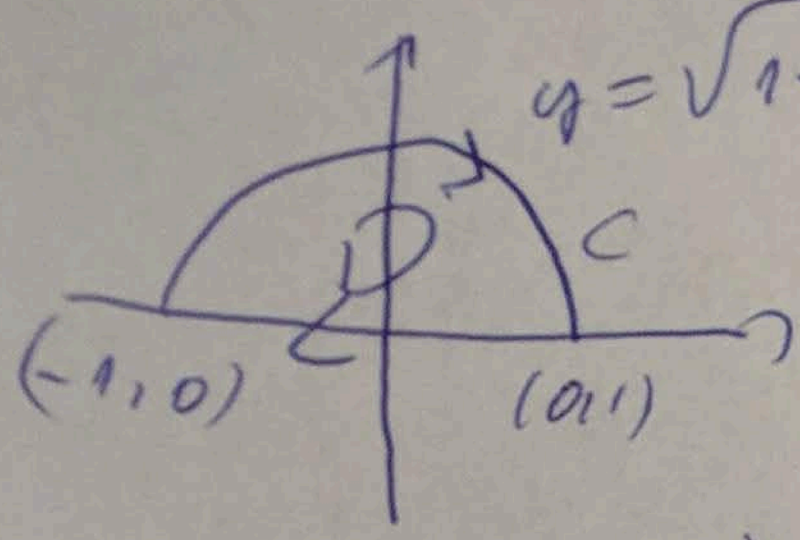
$D = \{(x,y) : 0 \le x \le 1, \; x \le y \le 1\}$

$= \{(x,y) : 0 \le y \le 1, \; 0 \le x \le y\}$

$\int_0^1 \int_x^1 \cos\left(\frac{x}{y}\right) dy\,dx \overset{(3)}{=} \int_0^1 \int_0^y \cos\left(\frac{t}{y}\right) dx\,dy =$

$= \int_0^1 \left[y \sin\left(\frac{x}{y}\right)\right]_{x=0}^{y} dy \overset{(2)}{=} \int_0^1 y \sin(1)\,dy = \frac{y^2}{2} \cdot \sin(1).$ (2)

(3) Sol:



Work: $W = \int_C \vec F\, d\vec r$

$\vec F(x,y) = 2x\vec i + (x^2 + 3xy^2)\vec j = P\vec i + Q\vec j$

Green: $\int_C \vec F\, d\vec r = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ positive orientation!

$W = \int_C \vec F\, d\vec r \overset{\text{negative or.}}{=} \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA$

$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 2x + 3y^2$ (1)

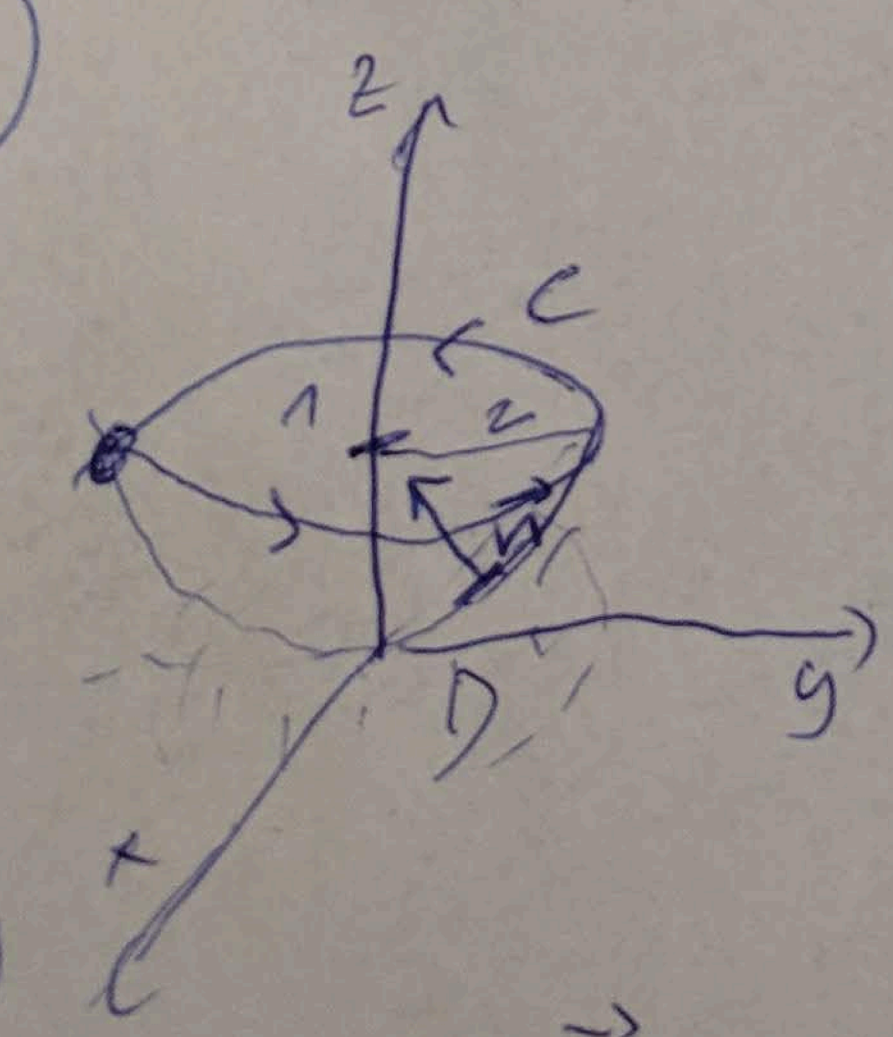
$D = \{(r\cos\varphi, r\sin\varphi) : 0 \le r \le 1,\ 0 \le \varphi \le \pi\}$

$W = \int_0^\pi \int_0^1 (\cancel{2r\cos\varphi} + 3r^2\sin^2\varphi)\, r\, dr\, d\varphi$ (3)

$= \int_0^\pi \sin^2\varphi\, d\varphi \int_0^1 r^3\, dr = \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2\varphi\right) d\varphi \cdot \frac{1}{4}$

$= \frac{\pi}{2} \cdot \frac{1}{4} = \boxed{\frac{\pi}{8}}$ (3)

(4)



Stokes: $\int_C \vec F\, d\vec r = \iint_S \operatorname{curl} \vec F \cdot d\vec S$

$F(x,y,z) = y^2 z\,\vec i - 2y\,\vec j + x\,\vec k$

(a) C:

$\vec r(t) = 2\cos t\,\vec i + 2\sin t\,\vec j + \vec k \qquad 0 \le t \le 2\pi$

$\vec r\,'(t) = -2\sin t\,\vec i + 2\cos t\,\vec j$

$\int_C \vec F\, d\vec r = \int_0^{2\pi} \vec F(\vec r(t)) \cdot \vec r\,'(t)\, dt = \int_0^{2\pi} (4\sin^2 t\,\vec i - 4\sin t\,\vec j) \cdot (-2\sin t\,\vec i + 2\cos t\,\vec j)\, dt$ (2)

$$= \int_0^{2\pi} -8\sin^3 t - \cancel{2\cos t \sin t}\, dt = -8\int_0^{2\pi} \sin t\,(1-\cos^2 t)\, dt$$

$$= \cancel{-8\int_0^{2\pi} \sin t\, dt} + 8\int_0^{2\pi} \sin t \cos^2 t\, dt = 8\left[-\frac{\cos^3 t}{3}\right]_0^{2\pi} = 0 \quad (2)$$

(b)

$$S: \; z = g(x,y) = \frac{1}{4}(x^2+y^2) \qquad (x,y) \in D = \{(x,y):\; x^2+y^2 \le 4\}$$

$$r_x(x,y) \times r_y(x,y) = -\frac{x}{2}\vec{i} - \frac{y}{2}\vec{j} + \vec{k} \qquad \text{(upward normal).}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & -2y & x \end{vmatrix} = \vec{i}(0-0) - \vec{j}(1-y^2) + \vec{k}(0-2yz)$$

$$= (1-y^2)\vec{j} - 2yz\,\vec{k} \quad (1)$$

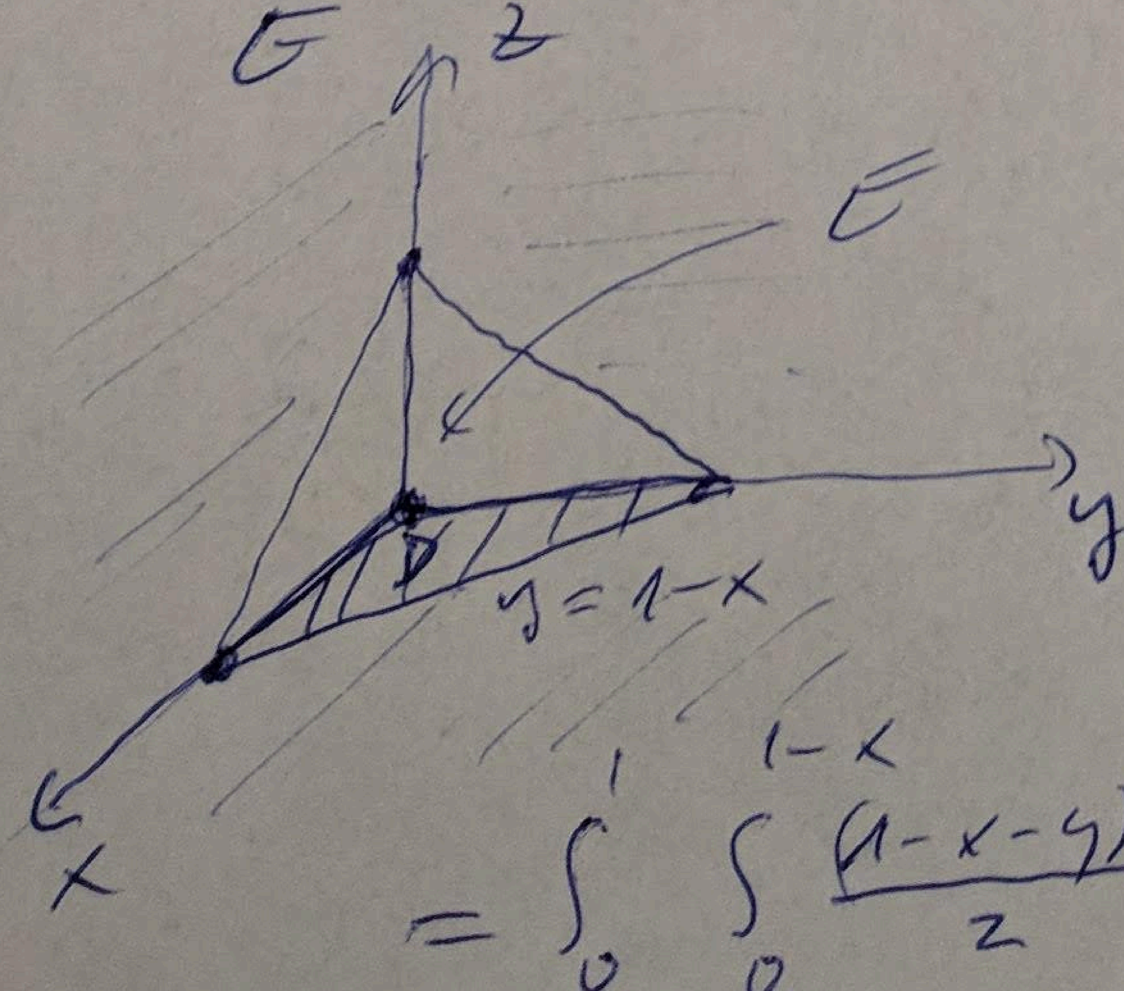
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D \left((1-y^2)\vec{j} - 2y\left(\frac{1}{4}x^2+\frac{1}{4}y^2\right)\vec{k}\right)\cdot\left(-\frac{x}{2}\vec{i} - \frac{y}{2}\vec{j} + \vec{k}\right) dA \quad (2)$$

$$= \iint_D (y^2-1)\frac{y}{2} - \frac{1}{2}y(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^{2} \left((r^2\sin^2\varphi - 1)\frac{r\sin\varphi}{2} - \frac{1}{2} r\sin\varphi \cdot r^2\right) r \, dr\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{2} \frac{r^4}{2} \cdot \sin^3\varphi \, dr\, d\varphi = 0. \quad (1)$$

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

5 Sol: $\iiint_E z\, dV$



$$E: (x,y) \in D:\; 0 \le z \le 1-x-y$$

$$D: 0 \le x \le 1 \qquad 0 \le y \le 1-x$$

$$\Rightarrow \iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx \quad (4)$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy\, dx =$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{(1-x-y)^3}{3} \right]_{y=0}^{1-x} dx = \frac{1}{6} \int_0^1 (1-x)^3 dx$$

$$= \frac{1}{6} \cdot \left[-\frac{(1-x)^4}{4} \right]_0^1 = \frac{1}{24}. \quad (4)$$

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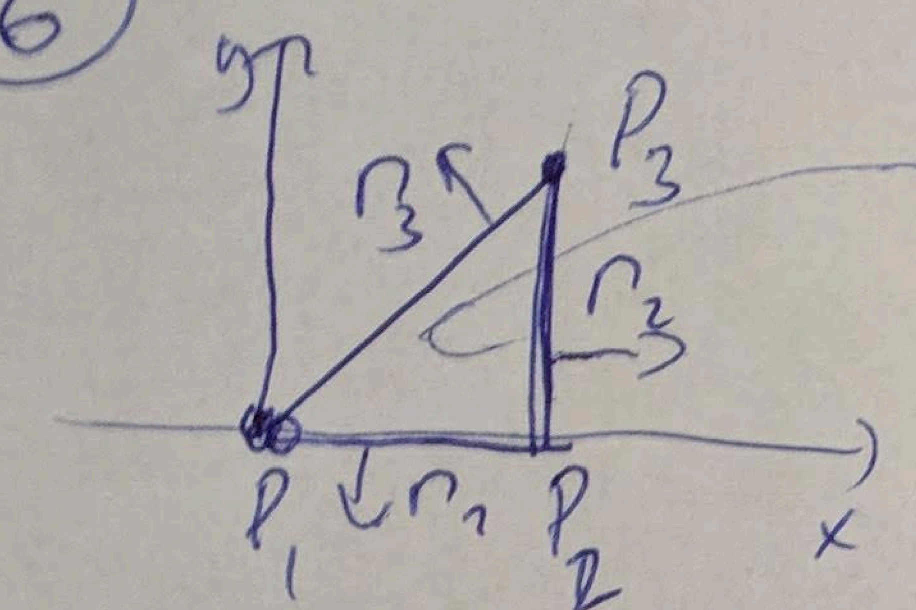

(a) $\lambda(x,y) = 1+x+y \qquad f(x,y) = 2 \qquad u_A = 1$

$$-\nabla \cdot \left((1+x+y) \cdot \nabla u(x,y)\right) = \left(-\frac{\partial}{\partial x}\vec{i} - \frac{\partial}{\partial y}\vec{j}\right) \cdot \left((1+x+y)\frac{\partial u}{\partial x}\vec{i} + (1+x+y)\frac{\partial u}{\partial y}\vec{j}\right)$$

$$= -(1+x+y)\frac{\partial^2 u}{\partial x^2} - (1+x+y)\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 2$$

Boundary: Γ_1: $\kappa = 1 \qquad \vec{n} = -\vec{j} \qquad y = 0 \qquad 0 < x < 1$

$$D_n u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)(-\vec{j}) = -\frac{\partial u}{\partial y}$$

$$\lambda D_n u + \kappa(u - u_A) = -(1+x)\frac{\partial u}{\partial y}(x,0) + 1(u(x,0) - 1) = 0$$

Γ_2: $\kappa = 2 \qquad n = \vec{i} \qquad x = 1 \qquad 0 < y < 1$

$$D_n u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right) \cdot \vec{i} = \frac{\partial u}{\partial x}$$

$$\lambda D_n u + \kappa(u - u_A) = (2+y)\frac{\partial u}{\partial x}(1,y) + 2(u(1,y) - 1) = 0$$

Γ_3: $\kappa = 0 \qquad \vec{n} = \frac{1}{\sqrt{2}}(-\vec{i} + \vec{j}) \qquad x = x \qquad y = x \qquad 0 < x < 1$

$$D_n u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\left(\frac{1}{\sqrt{2}} - \vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = -\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}$$

$$\lambda D_n u + \kappa(u - u_A) = (1+2x)\left(-\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x}(x,x) + \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}(x,x)\right) = 0 \Rightarrow \frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(x,x)$$

Boundary value problem:

Find $u = u(x,y)$ such that:

$$-(1+x+y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 2 \qquad (x,y)\in\Omega \qquad (1)$$

$$-(1+x)\frac{\partial u}{\partial y}(x,0) + u(x,0) - 1 = 0 \qquad 0<x<1 \qquad (1)$$

$$(2+y)\frac{\partial u}{\partial x}(1,y) + 2(u(1,y)-1) = 0 \qquad 0<y<1 \qquad (1)$$

$$\frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(\cancel{x},x) \qquad 0<x<1. \qquad (1)$$

(b) General weak formulation:

$$\iint_\Omega \kappa \nabla u\cdot\nabla v\,dA + \int_\Gamma \gamma u v\,ds = \iint_\Omega f v\,dA + \int_\Gamma (\overset{0}{\overset{\|}{g}} + \gamma u_A) v\,ds$$

Ω: $0<x<1 \qquad 0<y<x$

$$\int_{\Gamma_1} h\,ds = \left| \begin{matrix} \vec{r}_1(t) = t\vec{i} + 0\vec{j} \\ |\vec{r}_1{}'(t)| = 1 \end{matrix} \right| = \int_0^1 h(t,0)\,dt$$

$$\int_{\Gamma_2} h\,ds = \left| \begin{matrix} \vec{r}_2(t) = \vec{i} + t\vec{j} \\ |\vec{r}_2{}'(t)| = 1 \end{matrix} \right| = \int_0^1 h(1,t)\,dt$$

Weak formulation: Find $u = u(x,y)$ such that

$$\int_0^1\int_0^x \overset{(1)}{(1+x+y)} \nabla u\cdot\nabla v\,dy\,dx + \overbrace{\int_0^1 u(t,0)v(t,0)\,dt + 2\int_0^1 u(1,t)v(1,t)\,dt}^{(1)}$$

$$= \int_0^1\int_0^x \overset{(1)}{2v}\,dy\,dx + \underbrace{\int_0^1 v(t,0)\,dt + 2\int_0^1 v(1,t)\,dt}_{(1)}$$

for all test functions v.

(c) $\phi_1(x,y) = ax+by+c$

$$\left.\begin{matrix} \phi_1(0,0) = 1 = c \\ \phi_1(1,0) = 0 = a+c \\ \phi_1(1,1) = 0 = a+b+c \end{matrix}\right\} \begin{matrix} c=1 \\ a=-1 \\ b=0 \end{matrix} \Rightarrow \phi_1(x,y) = 1-x \qquad (1)$$

$\phi_2(x,y) = ax+by+c$

$$\left.\begin{aligned} \phi_2(0,0) &= 0 = c \\ \phi_2(1,0) &= 1 = a+c \\ \phi_2(1,1) &= 0 = a+b+c \end{aligned}\right\} \begin{aligned} c &= 0 \\ a &= 1 \\ b &= -1 \end{aligned} \Rightarrow \phi_2(x,y) = x - y \quad (1)$$

$$\phi_3(x,y) = ax + by + c$$

$$\left.\begin{aligned} \phi_3(0,0) &= 0 = c \\ \phi_3(1,0) &= 0 = a+c \\ \phi_3(1,1) &= 1 = a+b+c \end{aligned}\right\} \begin{aligned} c &= 0 \\ a &= 0 \\ b &= 1 \end{aligned} \Rightarrow \phi_3(x,y) = y \quad (1)$$

(d)

$$b_2 = \int_0^1 \int_0^x 2(x-y)\,dy\,dx + \int_0^1 t\,dt + 2\int_0^1 (1-t)\,dt = \quad (1)$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx + \frac{1}{2} + 2\cdot\frac{1}{2} = 2\int_0^1 \frac{x^2}{2}\,dx + \frac{3}{2} = \frac{1}{3} + \frac{3}{2} = \frac{2+9}{6} = \boxed{\frac{11}{6}} \quad (1)$$