MVE515 Computational Mathematics

FEM1: Boundary value problems and finite elements in 1D

1

Introduction

In this course we will study *partial differential equations* (PDEs); that is equations that contain the partial derivatives of the unknown multivariate function $u = u(x, y, z)$. For example, the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and Poisson's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z).$$

A PDE is set up in a domain in 3-dimensional space, $D \subset \mathbf{R}^3$, or on the pale, $D \subset \mathbf{R}^2$. In order to obtain a unique solution one requires *boundary conditions* specified on the boundary $S = \partial D$ of the domain D, for example,

$$u = 0 \quad \text{on } S.$$

We call this a *boundary value problem.*

We will also consider time-dependent PDEs, for example, the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

and wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0.$$

PDEs show up in many areas: heat conduction, diffusion, wave phenomenon, electromagnetic fields, kvantum mechanics, fluid dynamics and so on. Despite the diverse application backgorund, these PDEs have many common features. Notice, for example, that the Laplace-operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

appears in all of the above examples.

We will solve these PDEs using the finite element method (FEM). This method was developed in the 1950s to solve PDEs appearing in on a computer civil engineering (construction of buildings, for example). The FEM is based on the so called *weak formulation* of the PDE.

To introduce the concepts, we first consider boundary value problems and FEM in 1D; that is, when $u = u(x)$, $x \in I \subset \mathbf{R}$.

We will use SI units, for example, `[K]` for temperature in Kelvin, `[m]` for length in meter, `[J]` for energy in Joule.

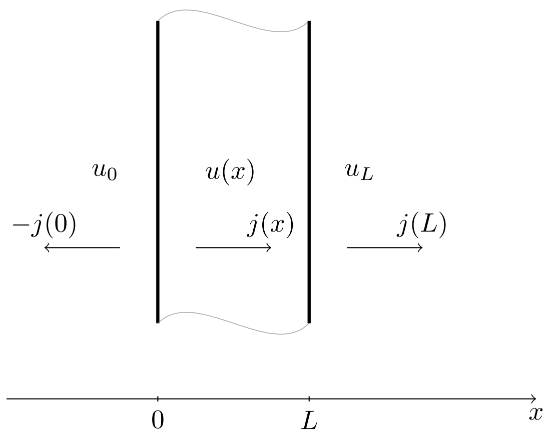

Figure 1: Heat conduction in a plate.

1.1 Boundary value problems in one variable

1.1.1 The heat equation

The stationary temperature $u(x)$ `[K]` at cross section x `[m]` of a plate with width L `[m]`, see Figure 1, satisfies differential equation

$$\mathrm{D}\big(-a(x)\,\mathrm{D}u(x)\big) = f(x), \quad x \in I = (0, L). \tag{1}$$

Here:

- $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}x}\ \left[\dfrac{1}{\mathtt{m}}\right]$ is the derivative;
- $f(x)\ \left[\dfrac{\mathtt{J}}{\mathtt{m}^3\,\mathtt{s}}\right]$ is the heat flux density of the source;
- $u(x)$ `[K]` temperature;
- $a(x)\ \left[\dfrac{\mathtt{J}}{\mathtt{m\,K\,s}}\right]$ is the thermal diffusivity;
- $j(x) = -a(x)\,\mathrm{D}u(x)\ \left[\dfrac{\mathtt{J}}{\mathtt{m}^2\,\mathtt{s}}\right]$ is the heat flux density the x-direction (Fourier's law).

We suppose that no quantity depends on the coordinates y and z. The same equation also describes heat conduction in a small rod with length L.

Dimension controll: the units equal on both sides of equation (1).

1.1.2 Boundary conditions

At $x = L$ it holds that the heat flux in the outward direction is proportional to the temperature difference:

$$j(L) = k_L\big(u(L) - u_L\big), \tag{2}$$

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[1] September 1, 2017, Mihály Kovács, Matematiska vetenskaper, Chalmers tekniska högskola

- u_L [K] is the temperature of the environment (ambient tempreture);
- $u(L)$ [K] is the temperature of the plate at the right boundary section;
- k_L $\left[\dfrac{\mathtt{J}}{\mathtt{m}^2\,\mathtt{K}\,\mathtt{s}}\right]$ is the heat transfer coefficient.

On the other hand the heat flux satisfies Fourier's law:

$$j(L) = -a(L)\,\mathrm{D}u(L).$$

Therefore,

$$-a(L)\,\mathrm{D}u(L) = k_L\big(u(L) - u_L\big),$$

and hence

$$a\,\mathrm{D}u + k_L\big(u - u_L\big) = 0 \quad \text{för } x = L.$$

At $x = 0$ it holds that the heat flux in the outward direction is proportional to the temperature difference:

$$-j(0) = k_0\big(u(0) - u_0\big), \tag{3}$$

as $-j(0)$ is the heat flux in the $-x$-direction (outward). Again, by Fourier's law we have $j(0) = -a(0)\,\mathrm{D}u(0)$. Thus,

$$a(0)\,\mathrm{D}u(0) = k_0\big(u(0) - u_0\big).$$

In summary, we write:

$$a\,\mathrm{D_N}u + k\big(u - u_\mathrm{A}\big) = 0 \quad \text{for } x = 0, L. \tag{4}$$

Here u_A is the ambient temperature; that is, $u_\mathrm{A} = u_0$ respectively $u_\mathrm{A} = u_L$, the coefficient is $k = k_0$ respectively $k = k_L$, and $\mathrm{D_N}$ is the directional derivative in the outward direction (outward normal direction); that is,

$$\mathrm{D_N} = -\frac{\mathrm{d}}{\mathrm{d}x} \text{ at } x = 0, \quad \mathrm{D_N} = \frac{\mathrm{d}}{\mathrm{d}x} \text{ at } x = L.$$

The coefficient k depends on how well the plate is isolated at the boundary.

Special case 1: $k = \infty$, no isolation. We divide by k,

$$\frac{1}{k}a\,\mathrm{D_N}u + u - u_\mathrm{A} = 0,$$

and let $k \to \infty$ to get $0 + u - u_\mathrm{A} = 0$. Thus

$$u = u_\mathrm{A} \quad \text{at } x = 0 \text{ or } x = L.$$

This boundary condition holds at the boundary that is not isolated, that is, at $x = 0$ or $x = L$. In this case the temperature is the same on the inside and the outside of the boundary.

Special case 2: $k = 0$, perfect isolation. With $k = 0$ we get

$$a\,\mathrm{D_N}u = 0,$$

that is, there is no heat flow at the isolated boundary.

As $a > 0$ one arrives at

$$\mathrm{D_N}u = 0 \quad \text{at } x = 0 \text{ or } x = L.$$

1.1.3 The boundary value problem

Equations (1) and (4) results in a boundary problem.

Boundary value problem. Find $u = u(x)$ such that

$$\begin{aligned} -\,\mathrm{D}\big(a\,\mathrm{D}u\big) &= f && \text{for } x \in I = (0, L), \\ a\,\mathrm{D}_{\mathrm{N}}u + k\big(u - u_{\mathrm{A}}\big) &= g && \text{for } x = 0, L. \end{aligned} \tag{5}$$

Here we added a prescribed inflow of heat with heat flux density g, with $g = g_0$ respectively $g = g_L$, see Figure 2. In this case one has to add the terms $-g_L$ respectively $-g_0$ to the right hand side of (2) and (3), respectively.

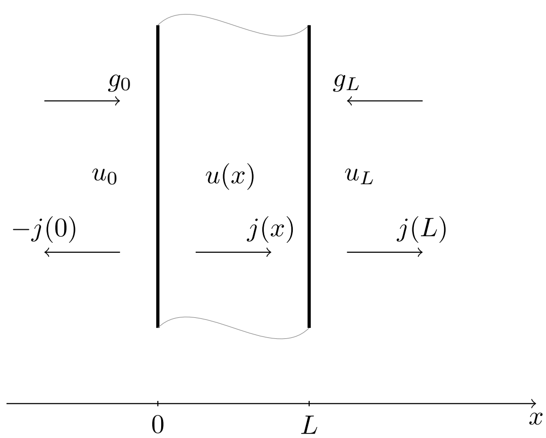

Figure 2: Heat conduction in a plate (with heat inflow at the boundary).

Boundary value problems can be solved, in principle, by integrating the differential equation twice and then specifying the integration constants using the two boundary conditions. However, very often this calculation cannot be performed analytically. This is when one employs FEM.

Example 1.1. A simple example is:

$$\begin{cases} -\mathrm{D}(5\mathrm{D}u) = 0, & \text{in } I = (0,1), \\ -5\mathrm{D}u(0) + 3(u(0) - 2) = 0, & \text{(isolation)} \\ u(1) = 0. & \text{(no isolation)} \end{cases}$$

Here we have $a = 5$, $f = 0$, $L = 1$, $k_0 = 3$, $u_0 = 2$, $g_0 = 0$, $k_1 = \infty$, $u_1 = 0$, $g_1 = 0$.

Integrating twice yields

$$\begin{aligned} -\,5\mathrm{D}u(x) &= C_1 \\ \mathrm{D}u(x) &= -\frac{1}{5}C_1 \\ u(x) &= -\frac{1}{5}C_1 x + C_2 \end{aligned}$$

Next, we use the boundary conditions to get

$$\begin{aligned} 0 &= -5\mathrm{D}u(0) + 3(u(0) - 2) = C_1 + 3C_2 - 6 \\ 0 &= u(1) = -\frac{1}{5}C_1 + C_2. \end{aligned}$$

We solve the system of equation with Gauss' elimination:

$$\begin{cases} C_1 + 3C_2 = 6, \\ -C_1 + 5C_2 = 0, \end{cases} \quad C_2 = \frac{3}{4},\ C_1 = \frac{15}{4}$$

The temperature and the heat flux are

$$u(x) = -\frac{3}{4}x + \frac{3}{4}, \quad j(x) = -a\mathrm{D}u(x) = -5\mathrm{D}u(x) = C_1 = \frac{15}{4}$$

1.2 Weak formulation

Here, we will rewrite the boundary value problem (5) to the so called weak form. We multiply the differential equation

$$-\mathrm{D}\big(a\,\mathrm{D}u\big) = f$$

with a function v and integrate by parts on $I = (0, L)$:

$$\begin{aligned} \int_0^L fv\,\mathrm{d}x &= -\int_0^L \mathrm{D}\big(a\,\mathrm{D}u\big)v\,\mathrm{d}x \\ &= -\Big[a\,\mathrm{D}u\,v\Big]_0^L + \int_0^L a\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x \\ &= a(0)\,\mathrm{D}u(0)\,v(0) - a(L)\,\mathrm{D}u(L)\,v(L) + \int_0^L a\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x. \end{aligned}$$

Now we use the boundary conditions from (5):

$$\begin{aligned} a(0)\,\mathrm{D}u(0) &= k_0\big(u(0) - u_0\big) - g_0, \\ -a(L)\,\mathrm{D}u(L) &= k_L\big(u(L) - u_L\big) - g_L. \end{aligned}$$

Therefore,

$$\int_0^L fv\,\mathrm{d}x = \Big(k_0\big(u(0) - u_0\big) - g_0\Big)v(0) + \Big(k_L\big(u(L) - u_L\big) - g_L\Big)v(L) + \int_0^L a\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x.$$

We collect the terms that involve the unknown function u on the left-hand side:

$$\begin{aligned} &\int_0^L a\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x + k_0u(0)v(0) + k_Lu(L)v(L) \\ &\qquad = \int_0^L fv\,\mathrm{d}x + \big(k_0u_0 + g_0\big)v(0) + \big(k_Lu_L + g_L\big)v(L). \end{aligned}$$

This equation must be fulfilled for every choice of a function v. The arbitrary function v is called *test function*. This equivalent formulation of the boundary problem is called the *weak formulation*. This is the basis for the finite element method.

The weak formulation. Find a fuction $u = u(x)$ such that the equation

$$\begin{aligned} &\int_0^L a\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x + k_0u(0)v(0) + k_Lu(L)v(L) \\ &\qquad = \int_0^L fv\,\mathrm{d}x + \big(k_0u_0 + g_0\big)v(0) + \big(k_Lu_L + g_L\big)v(L) \end{aligned} \tag{6}$$

holds for all test functions v.

(To get an entirely correct weak formulation one would have to specify the class of functions where u and v belongs to. We will not go into this in the present course.)

The boundary condition $u = u_{\mathrm{A}}$ ($k = \infty$) is a little special. In this case one has to choose test functions that satisfy $v = 0$ at the boundary where $u = u_{\mathrm{A}}$ is specified. Then, the corresponding terms in (6) is $v(0) = 0$ and/or $v(L) = 0$.

Example 1.2.

$$\begin{cases} -\mathrm{D}\big((1+x^2)\,\mathrm{D}u\big) = 1 \quad \text{in } (-1,1), \\ u(-1) = 0, \quad \mathrm{D}u(1) = 0. \end{cases}$$

(a) Write down the weak formulation of the problem.
(b) solve the problem by integrating it twice.
Solution. (a) Because of the boundary condition $u(-1) = 0$ we employ test functions v such that $v(-1) = 0$. We multiply with v and integrate:

$$\begin{aligned} \int_{-1}^{1} v\,\mathrm{d}x &= -\int_{-1}^{1} \mathrm{D}\big((1+x^2)\,\mathrm{D}u\big)\,v\,\mathrm{d}x \quad \big\{\text{integration by parts}\big\} \\ &= -\Big[(1+x^2)\,\mathrm{D}u(x)\,v(x)\Big]_{-1}^{1} + \int_{-1}^{1} (1+x^2)\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x \\ &= -2\,\underbrace{\mathrm{D}u(1)}_{=0}\,v(1) + 2\,\mathrm{D}u(-1)\,\underbrace{v(-1)}_{=0} + \int_{-1}^{1} (1+x^2)\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x \\ &= \int_{-1}^{1} (1+x^2)\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x. \end{aligned}$$

The weak formulation is: Find $u = u(x)$ such that $u(-1) = 0$ and

$$\int_{-1}^{1} (1+x^2)\,\mathrm{D}u\,\mathrm{D}v\,\mathrm{d}x = \int_{-1}^{1} v\,\mathrm{d}x \quad \text{for all } v \text{ with } v(-1) = 0.$$

(b) The differential equation reads as

$$\mathrm{D}\big((1+x^2)\,\mathrm{D}u\big) = -1$$

We integrate

$$\begin{aligned} (1+x^2)\,\mathrm{D}u &= -x + C, \\ \mathrm{D}u(x) &= -\frac{x}{1+x^2} + \frac{C}{1+x^2}, \\ u(x) &= -\tfrac{1}{2}\ln(1+x^2) + C\arctan(x) + D. \end{aligned}$$

The boundary conditions yield:

$$\begin{aligned} 0 &= u(-1) = -\tfrac{1}{2}\ln(2) + C\arctan(-1) + D = -\tfrac{1}{2}\ln(2) - C\frac{\pi}{4} + D, \\ 0 &= \mathrm{D}u(1) = -\tfrac{1}{2} + \tfrac{1}{2}C. \end{aligned}$$

We have that $C = 1$, $D = \frac{1}{2}\ln(2) + \frac{\pi}{4}$. Thus, the solution is

$$\begin{aligned} u(x) &= -\tfrac{1}{2}\ln(1+x^2) + \arctan(x) + \frac{1}{2}\ln(2) + \frac{\pi}{4} \\ &= \ln\left(\sqrt{\frac{2}{1+x^2}}\right) + \arctan(x) + \frac{\pi}{4}. \end{aligned}$$

You should learn how to write down the weak formulation of boundary value problems for all combinations of boundary conditions and also how to solve simple boundary value problems by integrating the equation twice (see, the exercises at the end of this note). But the most important thing is to be able to solve a general boundary value problem using the finite element method.

1.3 The finite element method in 1–D

We will compute an approximate solution $U(x)$ that is a *piecewise linear function.* Therefore we consider a mesh in the interval $I = (0, L)$:

$$0 = x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_N = L.$$

Note: we use "MATLAB-numbering" that starts with 1. We always consider N points (also called nodes) x_i and $N-1$ intervals $I_i = (x_i, x_{i+1})$ of length $h_i = x_{i+1} - x_i$. See Figure 3.

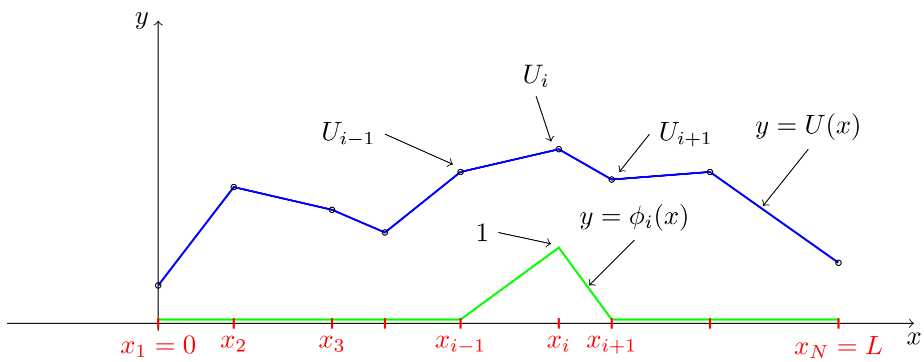

Figure 3: A piecewise linear function $y = U(x)$ and a basis function $y = \phi_i(x)$.

A continuous piecewise function $y = U(x)$ is completely determined by its nodal values $U_i = U(x_i)$. To represent U we will use *basis functions* $y = \phi_i(x)$, one for each node x_i.

The functions $y = \phi_i(x)$ are given the following way: they are continuous, piecewise linear functions such that

$$\phi_i(x_j) = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

See Figure 3. A general, continuous, piecewise linear function $y = U(x)$ can be uniquely written as a linear combination of basis functions:

$$U(x) = \sum_{i=1}^{N} U_i \phi_i(x), \quad \text{with coefficients } U_i = U(x_i).$$

Note that

$$U(x_j) = \sum_{i=1}^{N} U_i \phi_i(x_j) = U_j$$

as $\phi_i(x_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$ and therefore only one term (with $i = j$) remains in the sum.

We now have a formula that expresses a general, continuous, piecewise linear function $y = U(x)$ using its nodal values U_i. We aim now calculate the unknown nodal values U_i so that the function $y = U(x)$ is an approximate solution to the boundary value problem (5). To do so we will use the weak formulation (6):

$$\begin{aligned} &\int_0^L a \,\mathrm{D}u \,\mathrm{D}v \,\mathrm{d}x + k_0 u(0) v(0) + k_L u(L) v(L) \\ &\qquad = \int_0^L f v \,\mathrm{d}x + \big(k_0 u_0 + g_0\big) v(0) + \big(k_L u_L + g_L\big) v(L). \end{aligned}$$

We replace the solution u in the weak formulation with the ansatz $U(x) = \sum_{i=1}^{N} U_i \phi_i(x)$ and use test functions $v = \phi_j$. Then, we get

$$\begin{aligned}\sum_{i=1}^{N} U_i \int_0^L a\, \mathrm{D}\phi_i\, \mathrm{D}\phi_j\, \mathrm{d}x + k_0 U_1 \phi_j(0) + k_L U_N \phi_j(L) \\ = \int_0^L f\phi_j\, \mathrm{d}x + \big(k_0 u_0 + g_0\big)\phi_j(0) + \big(k_L u_L + g_L\big)\phi_j(L), \quad j = 1, \ldots, N.\end{aligned}$$

Using the notation

$$a_{ij} = a_{ji} = \int_0^L a\, \mathrm{D}\phi_i\, \mathrm{D}\phi_j\, \mathrm{d}x, \quad b_j = \int_0^L f\phi_j\, \mathrm{d}x,$$

and

(7) $\quad r_{11} = k_0, \quad r_{NN} = k_L, \quad r_{ij} = 0$, else; $s_1 = k_0 u_0 + g_0, \quad s_N = k_L u_L + g_L, \quad s_j = 0$, else,

we arrive at

$$\sum_{i=1}^{N} (a_{ij} + r_{ij}) U_i = b_j + s_j, \quad j = 1, \ldots, N;$$

or, in the matrix form:

$$(\mathcal{A} + \mathcal{R})\mathcal{U} = b + s.$$

This is a linear system of equations of N equations and N unknowns. The matrix $\mathcal{R}$ and the vector s are related to the boundary conditions.

The matrix $\mathcal{K} := \mathcal{A} + \mathcal{R}$ is called the *stiffness matrix*. The stiffness matrix is *symmetric*; that is, $k_{ji} = k_{ij}$, and *tridiagonal*,

$$\mathcal{K} = \begin{bmatrix} * & * & 0 & \ldots & 0 \\ * & * & * & \ddots & \vdots \\ 0 & * & * & * & 0 \\ \vdots & \ddots & * & * & * \\ 0 & \ldots & 0 & * & * \end{bmatrix},$$

that is, $k_{ij} = 0$ except for $j = i-1, i, i+1$. The vector $l := b + s$ is called the *load vector*. The interval $I_i = (x_i, x_{i+1})$ together with its two basis functions ϕ_i, ϕ_{i+1} is called a *finite element*.

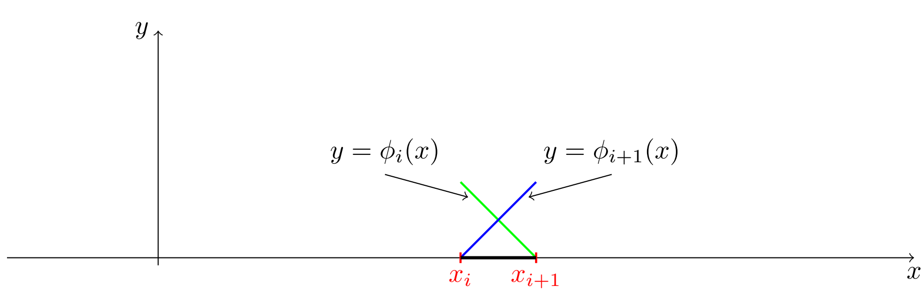

Figure 4: A finite element.

Time dependent problems: heat equation

Here we consider the time-dependent heat equation.

Initial-boundary value problem for the heat equation. Find $u = u(x,t)$ such that

$$
(8)\qquad
\begin{aligned}
D_t u(x,t) - \mathrm{D}_x\big(a(x)\,\mathrm{D}_x u(x,t)\big) &= f(x,t) \quad \text{for } x \in I = (0,L), \quad t > 0;\\
a\,\mathrm{D_N} u + k\big(u - u_{\mathrm{A}}(t)\big) &= g(t) \qquad\qquad \text{for } x = 0, L;\\
u(x,0) &= w(x) \quad \text{for } x \in I.
\end{aligned}
$$

We allow the heat flux density of the source (f) and on the boundary (g) as well as the ambient temperature (u_A) to change over time. The weak formulation (8) can be derived exactly the same way as for the stationary heat conduction problem: multiply the equation by a test function v, integrate by parts and use the boundary conditions. This yields:

The weak formulation of the heat equation. Find a function $u = u(x,t)$ such that $u(x,0) = w(x)$ and for all $t > 0$, the equation

$$
(9)\qquad
\begin{aligned}
&\int_0^L D_t u\, v\,\mathrm{d}x + \int_0^L a\,\mathrm{D}_x u\,\mathrm{D}_x v\,\mathrm{d}x + k_0 u(0,t)v(0) + k_L u(L,t)v(L)\\
&\qquad = \int_0^L f v\,\mathrm{d}x + \big(k_0 u_0(t) + g_0(t)\big)v(0) + \big(k_L u_L(t) + g_L(t)\big)v(L)
\end{aligned}
$$

holds for all test functions v.

The finite element approximation is again based on the weak formulation (9). We replace the solution u in the weak formulation with the ansatz $U(x,t) = \sum_{i=1}^{N} U_i(t)\phi_i(x)$ and use test functions $v = \phi_j$. This yields

$$
\begin{aligned}
&\sum_{i=1}^{N} \dot U_i(t) \int_0^L \phi_i\,\phi_j\,\mathrm{d}x + \sum_{i=1}^{N} U_i(t) \int_0^L a\,\mathrm{D}_x\phi_i\,\mathrm{D}_x\phi_j\,\mathrm{d}x + k_0 U_1(t)\phi_j(0) + k_L U_N(t)\phi_j(L)\\
&\quad = \int_0^L f(x,t)\phi_j(x)\,\mathrm{d}x + \big(k_0 u_0(t) + g_0(t)\big)\phi_j(0) + \big(k_L u_L(t) + g_L(t)\big)\phi_j(L), \quad j = 1,\dots,N.
\end{aligned}
$$

Using the notation

$$
a_{ij} = a_{ji} = \int_0^L a\,\mathrm{D}_x\phi_i\,\mathrm{D}_x\phi_j\,\mathrm{d}x, \quad m_{ij} = m_{ji} = \int_0^L \phi_i\,\phi_j\,\mathrm{d}x, \quad b_j(t) = \int_0^L f(x,t)\phi_j(x)\,\mathrm{d}x,
$$

and

$$
r_{11} = k_0, \quad r_{NN} = k_L, \quad r_{ij} = 0, \text{ else; } s_1(t) = k_0 u_0(t) + g_0(t), \quad s_N = k_L u_L(t) + g_L(t), \quad s_j = 0, \text{else},
$$

we arrive at

$$
\sum_{i=1}^{N} m_{ij}\dot U_i(t) + \sum_{i=1}^{N} (a_{ij} + r_{ij})U_i(t) = b_j(t) + s_j(t), \quad j = 1,\dots,N;
$$

or, in the matrix form:

$$
\mathcal{M}\,\dot U(t) + (\mathcal{A} + \mathcal{R})\,\mathcal{U}(t) = b(t) + s(t).
$$

or, as before, with $\mathcal{K} = \mathcal{A} + \mathcal{R}$ and $l(t) = b(t) + s(t)$

$$
\mathcal{M}\,\dot U(t) + \mathcal{K}\,\mathcal{U}(t) = l(t).
$$

This is a linear, first order differential equation system that could be solved by a time-stepping method, such as, the Backward Euler method. One needs to supplement this equation by an initial vector $U(0) = y$ obtained from the initial data as $y_j = \int_0^L w(x)\phi_j(x)\,\mathrm{d}x$. The matrix $\mathcal{M}$ is called the *mass matrix*.

Time dependent problems: wave equation

Here we consider the wave equation that can be used, for example, to describe the displacement u of a vibrating string of length L.

Initial-boundary value problem for the wave equation. Find $u = u(x,t)$ such that

$$
\begin{aligned}
D_t^2 u(x,t) - a^2\,\mathrm{D}_x^2 u(x,t) &= f(x,t) \quad \text{for } x \in I = (0,L), \quad t > 0; \\
\tau\,\mathrm{D_N} u + ku &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x = 0, L; \\
u(x,0) &= w(x) \quad \text{for } x \in I; \\
D_t u(x,0) &= z(x) \quad \text{for } x \in I.
\end{aligned}
\tag{10}
$$

Here $a = \frac{\rho}{\tau}$, where τ is the constant horizontal tension and ρ is the linear density of the string (assumed to be constant for simplicity). Here the boundary conditions model elastic attachment at the end of the strings; that is, when the end of the string is attached to springs with spring constants k_0 and k_L.

The weak formulation (10) can be derived exactly the same way as the heat equation: multiply the equation by a test function v, integrate by parts and use the boundary conditions:

The weak formulation of the wave equation. Find a function $u = u(x,t)$ such that $u(x,0) = w(x)$, $D_t u(x,0) = z(x)$ and, for all $t > 0$, the equation

$$
\begin{aligned}
&\int_0^L D_t^2 u\, v\,\mathrm{d}x + a^2 \int_0^L \mathrm{D}_x u\,\mathrm{D}_x v\,\mathrm{d}x + \frac{a^2 k_0}{\tau} u(0,t)v(0) + \frac{a^2 k_L}{\tau} u(L,t)v(L) \\
&\quad = \int_0^L f v\,\mathrm{d}x
\end{aligned}
\tag{11}
$$

holds for all test functions v.

To set up the finite element method we proceed as before and we replace the solution u in the weak formulation (11) with the ansatz $U(x,t) = \sum_{i=1}^N U_i(t)\phi_i(x)$ and use test functions $v = \phi_j$. This yields This yields

$$
\begin{aligned}
&\sum_{i=1}^N \ddot{U}_i(t) \int_0^L \phi_i\,\phi_j\,\mathrm{d}x + \sum_{i=1}^N U_i(t) a^2 \int_0^L \mathrm{D}_x\phi_i\,\mathrm{D}_x\phi_j\,\mathrm{d}x + \frac{a^2 k_0}{\tau} U_1(t)\phi_j(0) + \frac{a^2 k_L}{\tau} U_N(t)\phi_j(L) \\
&\quad = \int_0^L f(x,t)\phi_j(x)\,\mathrm{d}x, \quad j = 1, \dots, N.
\end{aligned}
$$

Using the notation

$$
a_{ij} = a_{ji} = a^2 \int_0^L \mathrm{D}_x\phi_i\,\mathrm{D}_x\phi_j\,\mathrm{d}x, \quad m_{ij} = m_{ji} = \int_0^L \phi_i\,\phi_j\,\mathrm{d}x, \quad b_j(t) = \int_0^L f(x,t)\phi_j(x)\,\mathrm{d}x,
$$

and

$$
r_{11} = \frac{a^2 k_0}{\tau}, \quad r_{NN} = \frac{a^2 k_L}{\tau}, \quad r_{ij} = 0, \text{ else;}
$$

we arrive at

$$\sum_{i=1}^{N} m_{ij} \ddot{U}_i(t) + \sum_{i=1}^{N} (a_{ij} + r_{ij}) U_i(t) = b_j(t), \quad j = 1, \dots, N;$$

or, in the matrix form:

$$\mathcal{M} \ddot{U}(t) + (\mathcal{A} + \mathcal{R}) \mathcal{U}(t) = b(t),$$

or, as before, with $\mathcal{K} = \mathcal{A} + \mathcal{R}$,

$$\mathcal{M} \ddot{U}(t) + \mathcal{K} \mathcal{U}(t) = b(t) \tag{12}$$

This is a second order, linear differential equation system that can be solved by first rewriting it as a first order system and using some time stepping method. One has to supplement (12) with two initial vectors $U(0) = y^1$ and $\dot{U}(0) = y^2$ obtained from the initial data as $y_j^1 = \int_0^L w(x) \phi_j(x)\, \mathrm{d}x$ and $y_j^2 = \int_0^L z(x) \phi_j(x)\, \mathrm{d}x$, $j = 1, \dots, N$.