MVE515 Computational Mathematics–Recommended Problem Set 1

Problems

Write down the weak formulation of the following boundary value problems.

Problem 1.1.

$$\begin{cases} -u'' = f \quad \text{in } (0,L), \\ -u'(0) + u(0) = g_0, \quad u(L) = 0. \end{cases}$$

Tips: $v(L) = 0$.

Problem 1.2.

$$\begin{cases} -u'' = f \quad \text{in } (0,L), \\ u(0) = 1, \quad u(L) = 2. \end{cases}$$

Tips: $v(0) = 0$ and $v(L) = 0$.

Problem 1.3.

$$\begin{cases} -u'' = f \quad \text{in } (0,L), \\ -u'(0) + 5u(0) = 1, \quad u'(L) = 1. \end{cases}$$

Problem 1.4.

$$\begin{cases} -(\cos(x)u'(x))' = x^2 \quad x \in (0,1), \\ u(0) = 0, \quad \cos(1)u'(1) + u(1) = 1 \end{cases}$$

Tips: $v(0) = 0$.

Solve the following boundary value problems by integrating twice.

Problem 1.5.

$$\begin{cases} -u'' = 1 \quad \text{in } (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Problem 1.6.

$$\begin{cases} -\mathrm{D}((1+x)\,\mathrm{D}u) = 2x \quad \text{in } (0,1), \\ -\mathrm{D}u(0) + u(0) = 0, \quad u(1) = \frac{3}{2}. \end{cases}$$

Problem 1.7.

$$\begin{cases} -\mathrm{D}(a\,\mathrm{D}u) = 0 \quad \text{in } (0,L) \text{ with constant } a > 0, \\ u(0) = u_0, \quad u(L) = u_L. \end{cases}$$

Calculate also the heat flux density $j = -a\mathrm{D}u$.

Problem 1.8.

$$\begin{cases} -au'' + u' = 1 \quad \text{in } (0,1) \text{ with constant } a > 0, \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Answers and solutions

1.1. Find $u = u(x)$ such that $u(L) = 0$ and

$$\int_0^L u'v'\,\mathrm{d}x + u(0)v(0) = \int_0^L fv\,\mathrm{d}x + g_0 v(0) \quad \text{for all } v \text{ with } v(L) = 0.$$

1.2. Find $u = u(x)$ such that $u(0) = 1$ and $u(L) = 2$ and

$$\int_0^L u'v'\,\mathrm{d}x = \int_0^L fv\,\mathrm{d}x \quad \text{for all } v \text{ with } v(0) = v(L) = 0.$$

1.3. Find $u = u(x)$ such that

$$\int_0^L u'v'\,\mathrm{d}x + 5u(0)v(0) = \int_0^L fv\,\mathrm{d}x + v(0) + v(L) \quad \text{for all } v.$$

1.4. Find $u = u(x)$ such that $u(0) = 0$ and

$$\int_0^1 \cos x\, u'(x)v'(x)\,\mathrm{d}x + u(1)v(1) = \int_0^1 x^2 v(x)\,\mathrm{d}x + v(1) \quad \text{for all } v \text{ with } v(0) = 0.$$

1.5. $u(x) = \frac{1}{2}x(1-x)$

1.6. $u(x) = -\frac{1}{2}x^2 + x + 1$

1.7. $u(x) = (u_L - u_0)\dfrac{x}{L} + u_0$, $j = -aj'(x) = a(u_0 - u_L)\dfrac{1}{L}$. Note that the heat flux density is constant and is proportional to the temperature difference.

1.8. One integration yields $u' - \frac{1}{a}u = -\frac{1}{a}(x + C)$. Multiply with the integrating factor $e^{-x/a}$ and integrate again. The solution is $u(x) = x - (e^{x/a} - 1)/(e^{1/a} - 1)$.