Problems

Problem 2.1. (a) Write down the boundary value problem for the 2D stationary heat equation on the unit square $(0,1) \times (0,1)$, with constant heat conductivity 3, constant heat source density equals 2, and constant ambient temperature equals 10 on all sides. On the boundary $y = 0$ the heat transfer coefficient equals 7, while on the rest of the boundary it is arbitrarily large. There is no prescribed heat influx at the boundary.

(b) Write down the weak formulation of the problem.

Problem 2.2. Same as 2.1 but heat conductivity equals $1 + xy$, no heat source, constant ambient temperature 10 on all sides, the heat transfer coefficient on the boundary $x = 0$ equals 0, it equals 7 on the boundary $x = 1$, and it is arbitrarily large on the boundaries $y = 0$ and $y = 1$.

Problem 2.3. (a) Write down the finite element basis functions ϕ_1, ϕ_2, ϕ_3 for a triangulation that consits of a single triangle T with nodes $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$.

(b) Compute the elements of the stiffness matrix

$$a_{ij} = \iint\limits_T \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A.$$

Problem 2.4. Write down the weak formulation of the following boundary value problems in 3D.
(a)

$$\begin{cases} -\nabla \cdot (a\nabla u) + cu = f & \text{in } \Omega, \\ u = u_{\mathrm{A}} & \text{on } \Gamma. \end{cases}$$

(b)

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = u_{\mathrm{A}} & \text{on } \Gamma_1, \\ \mathrm{D}_N u = g & \text{on } \Gamma_2, \end{cases}$$

where $\Delta u = \nabla \cdot \nabla u$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ (Γ_1 and Γ_2 are non-overlapping).

Solutions.

2.1. (a) On the boundary $y = 0$ we have $g = 0$ and the outward pointing normal vector is $\mathbf{n} = -\mathbf{j}$. Thus the boundary condition on this boundary is:

$$\begin{aligned}\lambda \mathrm{D}_N u + \kappa(u - u_\mathrm{A}) &= 3(-\mathbf{j}) \cdot \nabla u(x,0) + 7(u(x,0) - 10) \\ &= -3\frac{\partial u(x,0)}{\partial y} + 7(u(x,0) - 10) = 0, \quad 0 < x < 1.\end{aligned}$$

The boundary value problem is

$$\begin{cases} -3\Delta u(x,y) = 2 \quad \text{for } 0 < x < 1,\, 0 < y < 1, \\ u(x,1) = u(0,y) = u(1,y) = 10, \text{ for } 0 < x < 1, \text{respectively}, \ 0 < y < 1, \\ -3\dfrac{\partial u(x,0)}{\partial y} + 7(u(x,0) - 10) = 0, \text{ for } 0 < x < 1. \end{cases}$$

(b) Find $u = u(x,y)$ such that $u(x,1) = u(0,y) = u(1,y) = 10$ for $0 < x < 1$, respectively, $0 < y < 1$ and

$$\begin{aligned}3\int_0^1\int_0^1 \nabla u(x,y) \cdot \nabla v(x,y)\,\mathrm{d}x\,\mathrm{d}y + 7\int_0^1 u(x,0)v(x,0)\,\mathrm{d}x \\ = 2\int_0^1\int_0^1 v(x,y)\,\mathrm{d}x\,\mathrm{d}y + 70\int_0^1 v(x,0)\,\mathrm{d}x\end{aligned}$$

for all test functions $v = v(x,y)$ with $v(x,1) = v(0,y) = v(1,y) = 0$ for $0 < x < 1$, respectively, $0 < y < 1$.

2.2. (a) On the boundary $x = 0$ we have $\lambda = 1 + xy = 1$, $g = 0$, $\kappa = 0$ and the outward normal vector is $\mathbf{n} = -\mathbf{i}$. The boundary condition here is

$$\lambda \mathrm{D}_N u + \kappa(u - u_\mathrm{A}) = (-\mathbf{i}) \cdot \nabla u(0,y) = -\frac{\partial u(0,y)}{\partial x} = 0 \quad 0 < x < 1.$$

On the boundary $x = 1$ we have $\lambda = 1 + xy = 1 + y$, $g = 0$, $\kappa = 7$ and the outward normal vector is $\mathbf{n} = \mathbf{i}$. The boundary condition here is

$$\begin{aligned}\lambda \mathrm{D}_N u + \kappa(u - u_\mathrm{A}) &= (1 + y)\mathbf{i} \cdot \nabla u(1,y) + 7(u(1,y) - 10) \\ &= (1 + y)\frac{\partial u(1,y)}{\partial x} + 7(u(1,y) - 10) = 0, \quad 0 < y < 1.\end{aligned}$$

The boundary value problem is:

$$\begin{cases} -\nabla \cdot ((1 + xy)\nabla u(x,y)) = 0 \quad \text{for } 0 < x < 1,\, 0 < y < 1, \\ u(x,0) = u(x,1) = 10, \quad 0 < x < 1, \\ \dfrac{\partial u(0,y)}{\partial x} = 0, \quad 0 < y < 1, \\ (1 + y)\dfrac{\partial u(1,y)}{\partial x} + 7(u(1,y) - 10) = 0, \quad 0 < x < 1. \end{cases}$$

(b) Find $u = u(x,y)$ such that $u(x,0) = u(x,1) = 10$, for $0 < x < 1$ and

$$\int_0^1\int_0^1 (1 + xy)\nabla u(x,y) \cdot \nabla v(x,y)\,\mathrm{d}x\,\mathrm{d}y + 7\int_0^1 u(1,y)v(1,y)\,\mathrm{d}y = 70\int_0^1 v(1,y)\,\mathrm{d}y$$

for all test functions $v = v(x,y)$ with $v(x,0) = v(x,1) = 0$ for $0 < x < 1$.

2.3. (a) The basis functions are of the form $\phi(x, y) = a + bx + cy$ and they are equal to 1 at one of the nodes and and to 0 at the other two. We thus get

$$\phi_1(x, y) = 1 - x - y,\ \ \phi_2(x, y) = x,\ \ \phi_3(x, y) = y.$$

(b)

$$\nabla\phi_1(x, y) = -\mathbf{i} - \mathbf{j},\ \ \nabla\phi_2(x, y) = \mathbf{i},\ \ \nabla\phi_3(x, y) = \mathbf{j}.$$

$$\begin{aligned} a_{11} &= \iint_T \nabla\phi_1 \cdot \nabla\phi_1 \,\mathrm{d}x\,\mathrm{d}y = \iint_T 2\,\mathrm{d}x\,\mathrm{d}y = 2\,\mathrm{area}(T) = 1, \\ a_{12} &= \iint_T \nabla\phi_1 \cdot \nabla\phi_2 \,\mathrm{d}x\,\mathrm{d}y = \iint_T (-1)\,\mathrm{d}x\,\mathrm{d}y = -\,\mathrm{area}(T) = -\tfrac{1}{2} \end{aligned}$$

and so on. We obtain

$$\mathcal{A} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

2.4. (a) Find $u = u(x, y, z)$ such that $u = u_{\mathrm{A}}$ on Γ and

$$\iiint_\Omega (a\nabla u \cdot \nabla v + cuv)\,\mathrm{d}V = \iiint_\Omega fv\,\mathrm{d}V$$

for all test functions $v = v(x, y, z)$ with $v = 0$ on Γ.

(b) Find $u = u(x, y, z)$ such that $u = u_{\mathrm{A}}$ on Γ_1 and

$$\iiint_\Omega \nabla u \cdot \nabla v\,\mathrm{d}V = \iint_{\Gamma_2} gv\,\mathrm{d}S$$

for all test functions $v = v(x, y, z)$ with $v = 0$ on Γ_1.