(11)

(a) Let $v = v(x)$ be a test function with $v(0) = 0$. Then:

$$\int_0^1 (2x+1)v(x)\,dx = -\int_0^1 D[(x+1)Du(x)]v(x)\,dx$$

$$= \underbrace{-\left[(x+1)Du(x)v(x)\right]_0^1} + \int_0^1 (x+1)Du(x)Dv(x)\,dx \quad (*)$$

$$-\left[(x+1)Du(x)v(x)\right]_0^1 = -\left(2Du(1)v(1) - \underbrace{Du(0)\overset{0}{v(0)}}_{0}\right)$$

$$= -Du(1)v(1) = (2 - 3u(1))v(1)$$

From B.C: $Du(1) = 2 - 3(u(1))$

$(*)$

$$= 3u(1)v(1) - 2v(1) + \int_0^1 (x+1)Du(x)v(x)\,dx$$

Weak formulation:

Find $u = u(x)$ such that $u(0) = 1$ and

$$\int_0^1 (x+1)Du(x)Dv(x)\,dx + 3u(1)v(1) = \int_0^1 (2x+1)v(x)\,dx + 2v(1)$$

for all test functions v such that $v(0) = 0$.

(b)
$$-D(x+1)Du = 2x+1$$

$$-(x+1)Du = x^2 + x + c$$

$$Du = -\frac{x^2+x+c}{x+1} = -\frac{x(x+1)+c}{x+1} = -x - \frac{c}{x+1}$$

$$u(x) = -\frac{x^2}{2} - c\ln|x+1| + D = -\frac{x^2}{2} - c\ln(x+1) + D$$

$0 \le x \le 1$

$1 = u(0) = D$

$u'(1) + 3u(1) = -1 - \frac{c}{2} + 3\left(-\frac{1}{2} - c\ln 2 + 1\right) = 2$

$\Rightarrow -1 - \frac{c}{2} - \frac{3}{2} - 3c\ln 2 + 3 = 2$

$-c\left(\frac{1}{2} + 3\ln 2\right) + \frac{1}{2} = 2$

$$c = \frac{-3/2}{1/2 + 3\ln 2} = \frac{-3}{1 + 6\ln 2}$$

$\Rightarrow u(x) = -\frac{x^2}{2} - \frac{3}{1+6\ln 2}\ln(x+1) + 1$

(1.2) $\vec{F}$ is conservative if $\text{curl}\,\vec{F} = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & \sin z & y\cos z \end{vmatrix}$$

$= \vec{i}(\cos z - \cos z) - \vec{j}(0 - 0) + \vec{k}(0 - 0)$.

$\Rightarrow \vec{F}$ is conservative.

Let f be such that $Df = \vec{F}$:

$\frac{\partial f}{\partial x} = 1 \Rightarrow f = x + C(y,z)$

$\frac{\partial f}{\partial y} = y, \sin z$

$\frac{\partial f}{\partial z} = y \cdot \cos z$

$\Downarrow$

$\frac{\partial f}{\partial y} = \frac{\partial C}{\partial y} = \sin z$

$\Downarrow$

$C(y,z) = y\sin z + K(z)$

$\Downarrow$

$\frac{\partial f}{\partial z} = \frac{\partial C}{\partial z} = y \cdot \cos z + K'(z) = y\cos z$

$\Rightarrow K'(z) = 0 \Rightarrow K(z) = C$

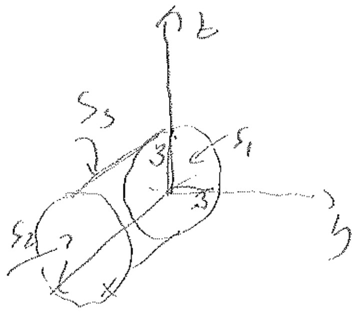
$\Rightarrow \boxed{f(x,y,z) = x + y \cdot \sin z + C}$

$$= \int_0^{2\pi}\int_0^{2} -4\cos^2\theta \cdot r^3 + -2r^3 + 10r \, dr\, d\theta$$

$$= 4\int_0^{2\pi} \underbrace{\cos^2\theta \, d\theta}_{1/2 + \frac{\cos 2\theta}{2}} \cdot \int_0^2 r^3 dr + \underbrace{\int_0^{2\pi}\int_0^2 -2r^3 + 10r \, dr\, d\theta}_{24\pi}$$

$$= 24\pi - 4\cdot\pi\cdot \frac{r^4}{4}\Big]_0^2 = 24\pi - 16\pi = \boxed{8\pi}$$

1.4



$$\vec{F} = x^2\vec{i} - y\vec{j} + z\vec{k}$$

$$\operatorname{div}\vec{F} = 2x - 1 + 1 = 2x$$

(a) $$\iiint_E \operatorname{div} F \, dV = \iint_{S_1}\int_0^2 2x \, dx \, dA = \iint_{S_1} x^2\Big]_0^2 dA$$

$$= \iint_{S_1} 4 \, dA = 4\cdot A(S_1) = 4\cdot 3^2\pi = 36\pi$$

(b) S_1: $y(u,v) = u\cdot\sin v$ $\quad z(u,v) = u\cdot\cos v$ $\quad x = 0$

$0 \le u \le 3$ $\quad 0 \le v \le 2\pi$

S_2: $y(u,v) = u\cdot\sin v$ $\quad z(u,v) = u\cdot\cos v$ $\quad x = 2$

S_3: $y = 3\sin v$ $\quad y = 3\cos v$ $\quad x = u$

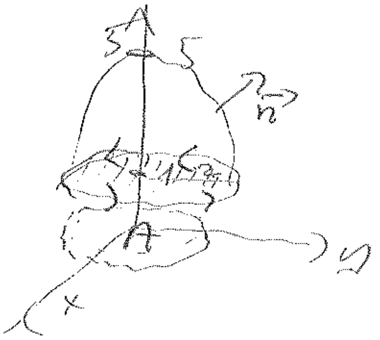
$0 \le u \le 2$ $\quad 0 \le v \le 2\pi$

(i) S_1: $\vec{r}_u = 0\vec{i} + \sin v\,\vec{j} + \cos v\,\vec{k}$

$\vec{r}_v = 0\vec{i} + u\cdot\cos v\,\vec{j} - u\cdot\sin v$

(1.3) To show: (5)

$$\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$



Positive orientation of C: anticlockwise when viewed from above.

when $z=1$: $1 = 5 - x^2 - y^2 \to x^2+y^2=4$

(a) C: $\vec{r}(t) = \underbrace{2\cdot\cos t}_{x(t)}\,\vec{i} + \underbrace{2\cdot\sin t}_{y(t)}\cdot\vec{j} + \underbrace{1}_{z(t)=1}\vec{k} \qquad 0 \le t \le 2\pi$

$$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi}\left(-2(2\sin t)\,\vec{i} + 2\sin t\,\vec{j} + 3\cdot 2\cos t\cdot\vec{k}\right)\cdot\left(-2\sin t\,\vec{i} + 2\cos t\cdot\vec{j}\right)dt$$

$$= \int_0^{2\pi} 8\sin^2 t + \overbrace{4\sin t\cos t}^{2\sin 2t}\, dt = \int_0^{2\pi} 8\sin^2 t\, dt + 0$$

$$= 8\int_0^{2\pi}\frac{1-\cos 2x}{2}\,dx = \int_0^{2\pi} 4\,dx = 8\pi.$$

(b) S: $x=x \quad y=y \quad 1 \le z \le 5 - x^2 - y^2 = g(x,y)$

$$\vec{n} = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} = +2x\,\vec{i} + 2y\,\vec{j} + \vec{k}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2yz & y & 3x \end{vmatrix} = \vec{i}(0) - \vec{j}(3+2y) + \vec{k}(0+2z)$$

$$= (-3-2y)\,\vec{j} + 2z\,\vec{k}$$

$$\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_A \left((-3-2y)\,\vec{j} + 2(-x^2-y^2+5)\,\vec{k}\right)\cdot\left(2x\,\vec{i} + 2y\,\vec{j} + \vec{k}\right) dA$$

$$= \iint_A -6y - 4y^2 - 2x^2 - 2y^2 + 10\, dA = \begin{vmatrix} x = r\cos\theta \\ y = r\sin\theta \end{vmatrix}$$

$4\cos^2\theta\cdot r^2$

$$= \int_0^{2\pi}\int_0^2 \left(-6r\cos\theta + 2r^2 - 10\right) r\, dr\, d\theta$$

$$\cancel{= 2\pi\int_0^2 2r^3 + 10r\, dr = 2\pi\left[\frac{r^4}{2} + 5r^2\right]_0^2}$$

$$\cancel{= 2\pi(-8+20) = 24\pi}$$

(4) S_1:

$x = 0 \qquad y(u,v) = u \cdot \cos v \qquad z = u \cdot \sin v$

$0 \leq u \leq 3$

$0 \leq v \leq 2\pi$

$\vec{r}_u = 0 \cdot \vec{i} + \cos v\, \vec{j} + \sin v\, \vec{k}$

$\vec{r}_v = 0 \vec{i} - u \sin v\, \vec{j} + u \cos v\, \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \cos v & \sin v \\ 0 & -u \sin v & u \cos v \end{vmatrix} = \vec{i} \cdot u$$

Outward normal : $-\vec{i} \cdot u$!

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{3} (0\vec{i} + -u\cos v \cdot \vec{j} + u \sin v \cdot \vec{k}) \cdot -\vec{i} \cdot u \, du\, dv$$

$$= 0$$

S_3: $x = 2 \qquad y = u \cos v \qquad z = u \sin v$

$\vec{r}_u = \cos v\, \vec{j} + \sin v\, \vec{k} \qquad \vec{r}_u \times \vec{r}_v = u \cdot \vec{i}$ ✓ outward

$\vec{r}_v = -u \sin v\, \vec{j} + u \cos v\, \vec{k}$

$$\iint_{S_2} \vec{F}\, d\vec{S} = \int_0^{2\pi} \int_0^{3} (4\vec{i} - u \cos v\, \vec{j} + u \sin v\, \vec{k}) \cdot u\vec{i} \, du\, dv$$

$$= \int_0^{2\pi} \int_0^{3} 4u \, du\, dv = 8\pi \cdot \frac{u^3}{2} \Big]_0^3 = 4\pi \cdot 9 = 36\pi$$

S_2: $x = u \qquad y = 3\cos v \qquad z = 3 \sin v \qquad$ $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$

$\vec{r}_u = \vec{i} + 0\vec{j} + 0\vec{k}$

$\vec{r}_v = 0\vec{i} + -3\sin v\, \vec{j} + 3\cos v\, \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -3\sin v & 3\cos v \end{vmatrix} = \vec{i}(0) - \vec{j}\, 3\cos v - k\, 3 \sin v$$
$$= -3\cos v\, \vec{j} - 3 \sin v\, \vec{k}$$

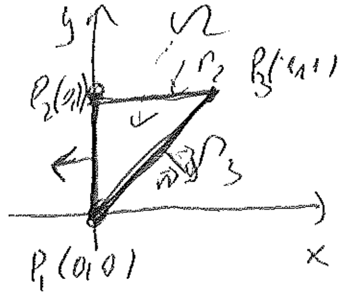
Outward normal: $3\cos v\, \vec{j} + 3 \sin v\, \vec{k}$

$$\iint_{S_2} \vec{F}\, d\vec{S} = \int_0^{2\pi}\int_0^{2} (u^2 \cdot \vec{i} - 3\cos v \cdot \vec{j} + 3\sin v \cdot \vec{k})(3\cos v\, \vec{j} + 3\sin v\, \vec{k})\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2} \underbrace{-9\cos^2 v + 9\sin^2 v}_{-9\cos(2v)}\, du\, dv = 0$$

$$\Rightarrow \iint_S \vec{F}\, d\vec{S} = \iiint_G \operatorname{div} \vec{F}\, dV$$

(1.5) (a)



$\lambda(x,y) = 3-x-y$

$f = 2$

$k(x,y) = y-x$

$u_A = 4$

$g = 0$

The equation:

$$-\nabla\cdot\Big((3-x-y)\,\nabla u(x,y)\Big) = 2 \qquad x,y\in\Omega$$

Boundary conditions:

Γ_1: $x=0$: unit normal: $-\vec{i}$

$$D_N u(0,y) = \nabla u\cdot\vec{n} = \left(\frac{\partial u}{\partial x}i + \frac{\partial u}{\partial y}\cdot j\right)(-i) = -\frac{\partial u(0,y)}{\partial x}$$

$$\lambda D_N u + k(u-u_A) = 0$$

$$(3-y)\frac{\partial u}{\partial x}(0,y) + y(u(0,y)-4) = 0 \qquad 0<y<1.$$

Γ_2: $y=1$. unit normal: $\vec{j}$.

$$D_N u = \nabla u\cdot\vec{n} = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\cdot\vec{j} = \frac{\partial u}{\partial y}$$

$$\lambda D_N u + k(u-u_A) = 0$$

$$(2-x)\frac{\partial u}{\partial y}(x,1) + (1-x)(u(x,1)-4) = 0 \qquad 0<x<1.$$

Γ_3: $x=y$ $\qquad \vec{n} = \frac{1}{\sqrt{2}}(\vec{i}-\vec{j})$

$$D_N u = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\left(\frac{1}{\sqrt{2}}(\vec{i}-\vec{j})\right) = \frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} - \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}$$

$$\lambda D_N u + k(u-u_A) = 0$$

Note! $k=0$ on Γ_3!

$$(3-2x)\left(\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x}(x,x) - \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}(x,x)\right) + 0\,(u(x,x)-4) = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(x,x).$$

Boundary value problem: (2)

Find $u = u(x,y)$ such that

$$-\nabla\cdot\left((3-x-y)\nabla u\right)$$

$$= -\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\left((3-x-y)\frac{\partial u}{\partial x}\vec{i} + (3-x-y)\frac{\partial u}{\partial y}\vec{j}\right)$$

$$= -\left(-\frac{\partial u}{\partial x} + \underbrace{(3-x-y)\frac{\partial^2 u}{\partial x^2}}_{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}} \underbrace{- \frac{\partial u}{\partial y} + (3-x-y)\frac{\partial^2 u}{\partial y^2}}_{\Delta u}\right)$$

$$\begin{cases} -(3-x-y)\Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2 & x \in \Omega. \\ (3-y)\frac{\partial u}{\partial x}(0,y) + y\left(u(0,y) - 4\right) = 0 & 0<y<1 \\ (2-x)\frac{\partial u}{\partial y}(1,x) + (1-x)\left(u(x,1) - 4\right) = 0 & 0<x<1 \\ \frac{\partial u}{\partial x}(x,x) = \frac{\partial u}{\partial y}(x,x) & 0<x<1 \end{cases}$$

(b) General weak formulation (No dirichlet boundary)

$$\iint_\Omega \lambda \nabla u \nabla v \, dA + \int_\Gamma \kappa u v \, ds = \iint_\Omega f v \, dA + \int_\Gamma (g + \kappa u_\infty) v \, ds$$

Ω: $0<y<1$: $0<x<y$.

Γ_1: $\vec{r}(t) = 0\vec{i} + t\vec{j}$ $\quad 0<t<1 \quad |r'(t)| = 1$

$\quad\quad \overset{\shortparallel}{x(t)} \quad y(t)$

$$\int_{\Gamma_1} u \, ds = \int_0^1 u(0,t)\sqrt{0^2+1^2}\, dt = \int_0^1 u(0,t)\, dt$$

Γ_2: $\vec{r}(t) = t\vec{i} + \vec{j}$ $\quad |r'(t)| = 1$

$$\int_{\Gamma_2} u \, ds = \int_0^1 u(t,1)\, dt$$

Γ_3: $\kappa = 0$ on Γ_3

Weak formulation:

Find $u = u(x,y)$ such that

$$-\int_0^1\int_0^y (3-x-y)\nabla u\cdot\nabla v\,dx\,dy + \int_0^1 t\,u(0,t)v(0,t)\,dt + \int_0^1 (1-t)u(t,1)v(t,1)\,dt$$
$$= \int_0^1\int_0^y 2v(x,y)\,dx\,dy + 4\int_0^1 t\,v(0,t)\,dt + 4\int_0^1 (1-t)v(t,1)\,dt$$

(c) ϕ_1: ~~[illegible]~~ $\phi_1(x,y) = ax+by+c$

$$\left.\begin{aligned}
&\phi_1(0,0) = 1 = c\\
&\phi_1(0,1) = 0 = b+c \Rightarrow b=-1\\
&\phi_1(1,1) = 0 = a+b+c \Rightarrow a=0
\end{aligned}\right\}\quad \begin{aligned}&\phi_1(x,y) = 1-y\\ &\nabla\phi_1 = -\vec{j}\end{aligned}$$

ϕ_2: $\phi_2(x,y) = ax+by+c$

$$\left.\begin{aligned}
&\phi_2(0,1) = 1 = b+c \Rightarrow b=1\\
&\phi_2(0,0) = 0 = c\\
&\phi_2(1,1) = 0 = a+b+c \Rightarrow a=-b=-1
\end{aligned}\right\}\quad \begin{aligned}&\phi_2(x,y) = y-x\\ &\nabla\phi_2 = \vec{j}-\vec{i}\end{aligned}$$

ϕ_3: $\phi_3(x,y) = ax+by+c$

$$\left.\begin{aligned}
&\phi_3(1,1) = 1 = a+b+c \Rightarrow a=1\\
&\phi_3(0,0) = 0 = c\\
&\phi_3(0,1) = \cancel{[illegible]}\ 0 = b+c \Rightarrow b=0
\end{aligned}\right\}\quad \begin{aligned}&\phi_3(x,y) = x\\ &\nabla\phi_3 = \vec{i}\end{aligned}$$

(d)
$$a_{11} = \overbrace{\int_0^1\int_0^y 1\,dx\,dy}^{\frac{1}{2}A(T)} + \int_0^1 t(1-t)(1-t)\,dt + \int_0^1 (1-t)\cdot 0\cdot 0\,dt$$
$$= \frac{1}{2} + \int_0^1 t(1-2t+t^2)\,dt = \frac{1}{2} + \int_0^1 t - 2t^2 + t^3\,dt = \frac{1}{2} + \left[\frac{t^2}{2} - \frac{2}{3}t^3 + \frac{t^4}{4}\right]_0^1$$
$$= \underbrace{\frac{1}{2} + \frac{1}{2}}_{1} - \frac{2}{3} + \frac{1}{4} = \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$$

$ds = -dt$

$$a_{22} = \iint 2\,dA + \int_0^1 t\cdot t\cdot t\,dt + \int_0^1 \overset{s}{(1-t)(1-t)(1-t)}\,dt$$

$$\int_0^1 t^3\,dt$$

$$= 2\cdot\frac{1}{2} + \left[\frac{t^4}{4}\right]_0^1 + \left[\frac{t^4}{4}\right]_0^1 = 1 + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{3}{2}}$$

$$a_{33} = \iint 1\,dA + \int_0^1 t\cdot 0\cdot 0\,dt + \underbrace{\int_0^1 (1-t)\cdot t\cdot t\,dt}_{\int_0^1 s(1-s)(1-s)\,ds} = \boxed{\frac{7}{12}}$$

$$= \frac{1}{12}$$

$$a_{31} = a_{13} = \iint 0\,dA + \int_0^1 t\cdot 0\,dt + \int_0^1 (1-t)\cdot 0\cdot t\,dt = \boxed{0}$$

$$a_{32} = a_{23} = \iint -1\,dA + \int_0^1 t\cdot 0\cdot t\,dt + \int_0^1 (1-t)\,t\,(1-t)\,dt = -\frac{1}{2} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

$$= -\frac{1}{4} - \frac{2}{3} = \frac{-3-8}{12} = \boxed{-\frac{11}{12}}$$

$$a_{12} = a_{21} = \iint -1\,dA + \int_0^1 t\,(1-t)\,t\,dt + \int_0^1 (1-t)\cdot 0\,(1-t)\,dt = \boxed{-\frac{11}{12}}$$

b_i

(e) $b_1: \int_0^1\int_0^y 2(1-y)\,dx\,dy = \int_0^1 2y(1-y)\,dy = 2\int_0^1 y - y^2\,dy = 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$

$$+\ 4\int_0^1 t(1-t)\,dt + 4\int_0^1 (1-t)\cdot 0\,dt = \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow \boxed{b_1 = 1}$$

$b_2: \int_0^1\int_0^y 2(y-x)\,dx\,dy = 2\int_0^1 \left[yx - \frac{x^2}{2}\right]_{x=0}^{y} dy = 2\int_0^1 y^2 - \frac{y^2}{2}\,dy = \int_0^1 y^2\,dy = \frac{1}{3}$

$$+ 4\int_0^1 t\cdot t\,dt + 4\int_0^1 (1-t)(1-t)\,dt = 4\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{8}{3} \quad \Big\}\ \boxed{b_2 = 3}$$

$b_3: \int_0^1\int_0^y 2x\,dx\,dy = \int_0^1 y^2\,dy = \frac{1}{3}$

$$+ 4\int_0^1 t\cdot 0\,dt + 4\int_0^1 (1-t)\cdot t\,dt = \frac{2}{3} \quad \Big\}\ \boxed{b_3 = 1}$$

(d) ⑤

$$a_{11} = \int_0^1\int_0^y (3-x-y)\cdot 1\,dx\,dy + \int_0^1 t(1-t)(1-t)\,dt + \int_0^1 (1-t)\cdot 0\cdot 0\,dt$$

$$= \int_0^1 3y - \underbrace{\frac{y^2}{2} - y^2}_{-3/2\,y^2}\,dy + \int_0^1 t-2t^2+t^3\,dt = \left[3\frac{y^2}{2} - \frac{y^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$

$$+ \underbrace{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}_{1/12} \qquad 3/4 - \frac{2}{3} \quad \frac{9-8}{12} \quad \frac{1}{12}$$

$$\Rightarrow a_{11} = \boxed{\frac{13}{12}}$$

$$a_{22} = \int_0^1\int_0^y (3-x-y)\cdot 2\,dx\,dy + \int_0^1 t\cdot t\cdot t\,dt + \int_0^1 (1-t)(1-t)(1\cdot t)\,dt$$

$$= 2 + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{5}{2}}$$

$$a_{33} = \int_0^1\int_0^y (3-x-y)\cdot 1\,dx\,dy + \int_0^1 t\cdot 0\cdot 0\,dt + \int_0^1 (1-t)\cdot t\cdot t\,dt =$$

$$= 1 + \frac{1}{12} = \boxed{\frac{13}{12}}$$

$$a_{31} = a_{13} = \iint 0\,dA + \int_0^1 t\cdot 0\,dt + \int_0^1 (1-t)\,0\cdot t\,dt = \boxed{0.}$$

$$a_{32} = a_{23} = \int_0^1\int_0^y -1(3-x-y)\,dx\,dy + \int_0^1 t\cdot 0\cdot t\,dt + \int_0^1 (1-t)\,t(1-t)\,dt$$

$$= -1 + \frac{1}{12} = \boxed{-\frac{11}{12}}$$

$$a_{12} = a_{21} = \int_0^1\int_0^y -1(3-x-y)\,dx\,dy + \int_0^1 t(1-t)t\,dt + \int_0^1 (1-t)\,0\,(1-t)\,dt = \boxed{-\frac{11}{12}}$$

(f)

$$\begin{bmatrix} \frac{13}{12} & -\frac{11}{12} & 0 \\ -\frac{11}{12} & \frac{5}{2} & -\frac{11}{12} \\ 0 & -\frac{11}{12} & \frac{13}{12} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

1.6 ~~Find $u = u(x,t)$ such that $u = \dots$ on Γ_1 $(t \geq 0)$ and for $t > 0$~~

We take a test !

1.6) We take a test function v such that $v=0$ on Γ_1. Multiply the equation by v and integrate.

(6)

$$\iiint_\Omega \partial_t u\, v\, dV - \iiint_\Omega \underbrace{\Delta u}_{\nabla\cdot(\nabla u)}\, v\, dV = \iiint_\Omega \partial_t u\, v\, dV - \left(\iint_\Gamma v\, \nabla u\cdot n\, dS - \iiint_\Omega \nabla u\cdot \nabla v\, dV\right)$$

$$\underset{\substack{\uparrow \\ v=0 \\ \text{on } \Gamma_1}}{=} \iiint_\Omega \partial_t u\, v\, dV + \iiint_\Omega \nabla u\cdot\nabla v\, dV - \iint_{\Gamma_2} v\, \underbrace{\partial_n u}_{g-u}\, dS$$

$$= \iiint_\Omega \partial_t u\, v\, dV + \iiint_\Omega \nabla u\cdot\nabla v\, dV - \iint_{\Gamma_2} v(g-u)\, dS$$

$$= \iiint_\Omega \partial_t u\, v\, dV + \iiint_\Omega \nabla u\cdot\nabla v\, dV - \iint_{\Gamma_2} v g\, dS + \iint_{\Gamma_2} u v\, dS$$

Weak formulation:

Find $u=u(x,y,z,t)$ such that $u(x,y,z,0)=u_0(x,y,z)$, $u(x,y,z,t)=u_A$ for $(x,y,z)\in\Gamma_1$ and $t>0$ and for $t>0$:

$$\iiint_\Omega \partial_t u\, v\, dV + \iiint_\Omega \nabla u\cdot\nabla v\, dV + \iint_{\Gamma_2} u v\, dS = \iint_{\Gamma_2} g v\, dS$$

for all test functions v such that $v=0$ on Γ_1.