1.1 Due to the Dirichlet boundary condition $u(L)=u_L$, multiply the differential equation by a test function v, with $v(L)=0$ and integrate on $(0,L)$:

$$\int_0^L fv\,dx = -\int_0^L u''v\,dx + \int_0^L bu'v\,dx$$

Now, use integration by parts in the first integral at the right side

$$-\int_0^L u''v\,dx = -u'v\Big|_0^L + \int_0^L u'v'\,dx = -u'(L)\underbrace{v(L)}_{=0} + \underbrace{u'(0)}_{=g_0}v(0) + \int_0^L u'v'\,dx.$$

Hence the weak formulation is: Find a function $u=u(x)$ such that $u(L)=u_L$ and

$$\int_0^L u'v'\,dx + \int_0^L bu'v\,dx = \int_0^L fv\,dx - g_0 v(0)$$

for all test functions v, with $v(L)=0$.

1.2 We have Dirichlet boundary condition at $x=2$, so we multiply the differential equation by a test function v, with $v(2)=0$. Then integrating over $(0,2)$ and integration by parts, we get

$$\int_0^2 xt\,v(x)\,dx = \int_0^2 D_t u(x,t)v(x)\,dx - \int_0^2 D_x\big((x-1)D_x u(x,t)\big)\cdot v(x)\,dx$$

$$= \int_0^2 D_t u(x,t)v(x)\,dx - \big[(x-1)D_x u(x,t)v(x)\big]_{x=0}^2 + \int_0^2 (x-1)D_x u(x,t)D_x v(x)\,dx$$

$$= \int_0^2 D_t u(x,t)v(x)\,dx - D_x u(2,t)\underbrace{v(2)}_{=0} \underbrace{- D_x u(0,t)}_{=3-u(0,t)}v(0) + \int_0^2 (x-1)D_x u(x,t)D_x v(x)\,dx$$

Hence the weak formulation is: Find a function $u=u(x,t)$ such that $u(x,0)=\cos x$, $u(2,t)=4$, and

$$\int_0^2 D_t u(x,t)v(x)\,dx + \int_0^2 (x-1)D_x u(x,t)D_x v(x)\,dx - u(0,t)v(0)$$

$$= \int_0^2 xt\,v(x)\,dx - 3v(0)$$

for all test functions v, with $v(2)=0$.

1.3 Integrating the differential equation once:

$$(x-3)\,Du = x^3+x^2+c \implies Du = \frac{x^3+x^2}{x-3} + \frac{c}{x-3}$$

Then dividing x^3+x^2 by $x-3$ we have

$$Du = x^2+4x+\frac{12x}{x-3} + \frac{c}{x-3} = x^2+4x+12\,\frac{x-3+3}{x-3} + \frac{c}{x-3}$$

$$\Rightarrow Du = x^2+4x+12+\frac{36+c}{x-3}$$

Integrate again and note that $x \in (0,1)$

$$u(x) = \frac{x^3}{3}+2x^2+12x+(36+c)\ln|x-3|+D$$

$$= \frac{x^3}{3}+2x^2+12x+(36+c)\ln(3-x)+D$$

Now use the boundary conditions

$$11 = Du(0) = 12+\frac{36+c}{-3} = -\frac{c}{3} \implies \boxed{c=-33}$$

$$\ln 8 = u(1) = \frac{1}{3}+14+(36-33)\ln 2 + D = \frac{43}{3}+\ln 8 + D \implies \boxed{D=-\frac{43}{3}}$$

Hence

$$u(x) = \frac{x^3}{3}+2x^2+12x+3\ln(3-x)-14$$

and

$$j(x) = -a(x)\,Du(x) = (x-3)\left(x^2+4x+12-\frac{3}{x-3}\right)$$

Note: To evaluate the integral $\int \frac{x^3+x^2}{x-3}\,dx$ we used the fact that

$$\frac{x^3+x^2}{x-3} = x^2+4x+\frac{12x}{x-3}$$

Another way to calculate this integral is, for example, change of variable.

$$\int \frac{x^3+x^2}{x-3}\,dx = \left\{\begin{matrix} t=x-3 \\ dt=dx \end{matrix}\right\} = \int \frac{(t+3)^3+(t+3)^2}{t}\,dt$$

$$= \int \frac{t^3+9t^2+27t+27+t^2+6t+9}{t}\,dt = \int t^2+10t+33+\frac{36}{t}\,dt$$

$$= \frac{t^3}{3}+5t^2+33t+36\ln|t|+D = \frac{(x-3)^3}{3}+5(x-3)^2+33(x-3)+36\ln(3-x)+D$$

1.4

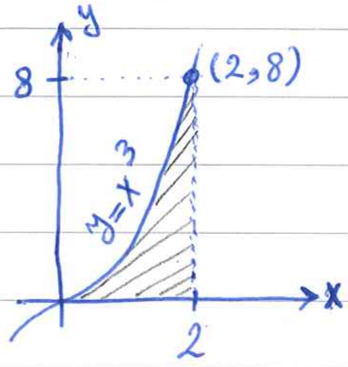
$$V=\iint_R \frac{x}{1+xy}\,dA=\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx=\int_0^1\Big[\ln(1+xy)\Big]_{y=0}^{y=1}dx$$

$$=\int_0^1 \ln(1+x)\,dx \ \underset{\text{by parts}}{\overset{\text{integration}}{=\!=\!=}}\ \Big((1+x)\ln(1+x)-(1+x)\Big]_{x=0}^{x=1}$$

$$=(2\ln 2-2)-(\ln 1-1)=2\ln 2-1$$

1.5 We note that the integral $\int e^{x^4}dx$ cannot be evaluated by elementary functions. So we change the order of integrals. Sketch of the domain

$$\begin{cases} 0\leq y\leq 8 \\ \sqrt[3]{y}\leq x\leq 2 \end{cases}$$



$$\int_0^8\int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy=\int_0^2\int_0^{x^3} e^{x^4}\,dy\,dx$$

$$=\int_0^2\Big[e^{x^4}y\Big]_{y=0}^{y=x^3}dx=\int_0^2 x^3e^{x^4}dx=\begin{Bmatrix} t=x^4 \\ dt=4x^3dx \end{Bmatrix}$$

$$=\frac{1}{4}e^{x^4}\Big]_{x=0}^{x=2}=\frac{e^{16}-1}{4}$$