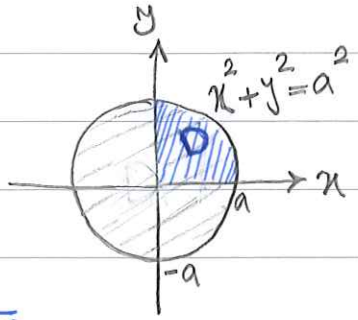
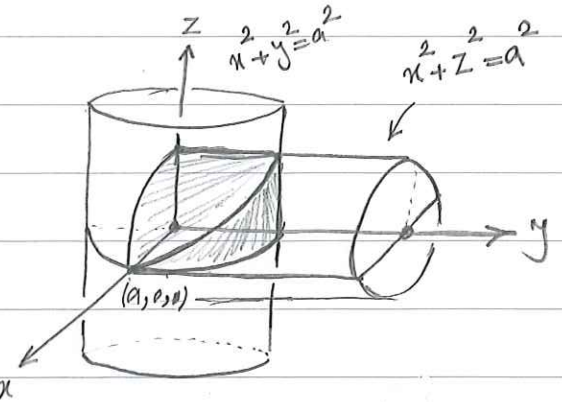
2.1



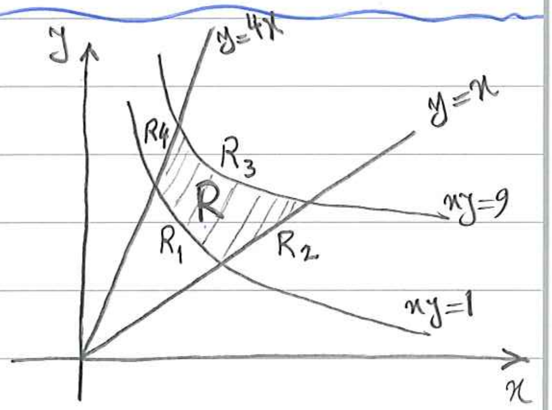


$$V=8\iint_D\int_0^{\sqrt{a^2-x^2}} dz\,dA$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx=8\int_0^a (a^2-x^2)\,dx=\underline{\frac{16}{3}a^3}$$

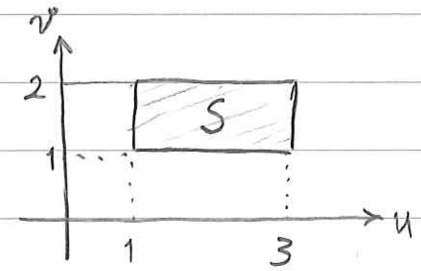
Note that we considered the first octant (to simplify), then multiply by 8.

2.2 $\quad V=\iint_R \sqrt{\frac{y}{x}}+\sqrt{xy}\,dA$



$$\begin{cases} x=\frac{u}{v} \\ y=uv \end{cases} \Rightarrow \begin{cases} u=\sqrt{xy} \\ v=\sqrt{\frac{y}{x}} \end{cases}$$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}=2\frac{u}{v}$$



$$\Rightarrow\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=2\frac{u}{v} \qquad (u>0,\ v>0)$$

$R_1: xy=1 \xrightarrow{T^{-1}} S_1: u=1$

$R_2: y=x \xrightarrow{T^{-1}} S_2: v=1$

$R_3: xy=9 \xrightarrow{T^{-1}} S_3: u=3$

$R_4: y=4x \xrightarrow{T^{-1}} S_4: v=2$

Hence

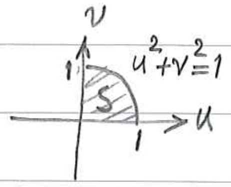
$$V=\iint_R \sqrt{\frac{y}{x}}+\sqrt{xy}\,dA=\iint_S (v+u)\left(2\frac{u}{v}\right)du\,dv$$

$$=\int_1^2\int_1^3\left(2u+\frac{2u^2}{v}\right)du\,dv=\underline{8+\frac{52}{3}\ln 2}$$

2.3 $\begin{cases} x = \frac{1}{3}u \\ y = \frac{1}{2}v \end{cases} \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{6}$

$R: 9x^2+4y^2=1\ ,\ x \geq 0,\ y \geq 0$

$\Rightarrow\ S: u^2+v^2=1\ ,\ u \geq 0\ ,\ v \geq 0$



$$\iint_R \sin(9x^2+4y^2)\,dA = \iint_S \sin(u^2+v^2)\left(\tfrac{1}{6}\right)du\,dv$$

$$\overset{\text{Polar coord}}{=} \frac{1}{6}\int_0^{\frac{\pi}{2}}\int_0^1 \sin(r^2)\,r\,dr\,d\theta$$

$$\underset{dt=2rdt}{\overset{t=r^2}{=}} \frac{1}{12}\int_0^{\frac{\pi}{2}} \left(-\cos r^2\right]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{12}\int_0^{\frac{\pi}{2}} (-\cos 1 + 1)\,d\theta = \boxed{\frac{(1-\cos 1)\pi}{24}}$$

Note: We could directly use $x = \frac{1}{3}r\cos\theta,\ y = \frac{1}{2}r\sin\theta$.

2.4 $\vec{r}(t) = \langle a\sin t, b\cos t, ct\rangle \Rightarrow \vec{a}(t) = \langle -a\sin t, -b\cos t, 0\rangle$

$\vec{F}(t) = m\vec{a}(t) = \langle -ma\sin t, -mb\cos t, 0\rangle$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(-ma^2\sin t\cos t + mb^2\sin t\cos t\right)dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{b^2-a^2}{2}m\,\sin 2t\,dt = \frac{b^2-a^2}{2}m\left(-\tfrac{1}{2}\right)\cos 2t\Big]_{t=0}^{\frac{\pi}{2}}$$

$$= \boxed{\frac{(b^2-a^2)m}{2}}$$

2.5 $\int_C \vec{r}\cdot d\vec{r} = \int_a^b \vec{r}(t)\cdot\vec{r}\,'(t)\,dt = \frac{1}{2}\int_a^b \frac{d}{dt}\left(|\vec{r}(t)|^2\right)dt$

$= \frac{1}{2}\left[|\vec{r}(t)|^2\right]_{t=a}^{t=b} = \frac{1}{2}\left[|\vec{r}(b)|^2 - |\vec{r}(a)|^2\right]$