Problem set 4.

Problem 4.1. Let $\mathbf{X} = \langle x, y, z \rangle$ and $\mathbf{F} = |\mathbf{X}|^2\mathbf{X}$. Show that $div\mathbf{F} = 5|\mathbf{X}|^2$ and then use the Divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where S is the sphere with radius R and center the origin.
Hint: Find $div\mathbf{F}$ either component wise or use $div(\phi\mathbf{G}) = \nabla\phi \cdot \mathbf{G} + \phi\ div\mathbf{G}$.
Note: If you cannot prove $div\mathbf{F} = 5|\mathbf{X}|^2$, you can use it to evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

Problem 4.2. Write the weak formulation for the BVP in 3D, on the solid region Ω enclosed in the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.

$$\begin{cases} -\nabla \cdot \left(e^{xyz}\nabla u\right) = x + 2y + 3z & \text{in } \Omega \\ u = 5 & \text{on } \Gamma_1 \quad (z = 0) \\ e^{xy}\nabla u \cdot \mathbf{n} = 2 & \text{on } \Gamma_2 \quad (z = 1) \\ e^{xyz}\nabla u \cdot \mathbf{n} + u = 3 & \text{on } \Gamma_3 \quad (x^2 + y^2 = 1) \end{cases}$$

Problem 4.3. (a) Write the finite element basis functions ϕ_1, ϕ_2, ϕ_3 for a triangulation that consists of a single triangle $T = \Omega$ with nodes $P_1 = (1, 0)$, $P_2 = (2, 0)$, $P_3 = (2, 1)$.

(b) Compute the elements of the stiffness matrix

$$a_{ij} = a_{ji} = \iint_\Omega \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \int_\Gamma \phi_i\phi_j \,\mathrm{d}s.$$

(c) Compute the elements of the mass matrix

$$m_{ij} = m_{ji} = \iint_\Omega \phi_i\phi_j \,\mathrm{d}A.$$