4.1

$$\text{div}\,\vec{F} = \text{div}\big(\underbrace{|\vec{X}|^2}_{\varphi}\underbrace{\vec{X}}_{\vec{G}}\big) = \nabla(|\vec{X}|^2)\cdot\vec{X} + |\vec{X}|^2\,\overbrace{\text{div}\,\vec{X}}^{=\nabla\cdot\langle x,y,z\rangle}$$

$$= \nabla(x^2+y^2+z^2)\cdot\langle x,y,z\rangle + (x^2+y^2+z^2)(1+1+1)$$

$$= (2x^2+2y^2+2z^2) + (3x^2+3y^2+3z^2) = 5(x^2+y^2+z^2) = 5|\vec{X}|^2$$

$S:\ x^2+y^2+z^2=R^2 \Rightarrow E:\ x^2+y^2+z^2 \leqslant R^2$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV = \iiint_E 5|\vec{X}|^2\,dV$$

Spherical coordinates $= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (5\rho^2)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$$= 5\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\varphi\, d\varphi\right)\left(\int_0^{R}\rho^4\, d\rho\right) = 5(2\pi)(2)\left(\frac{R^5}{5}\right) = \underline{4\pi R^5}$$

4.2

Multiply by a test function $v=v(x,y,z)$, integrate over Ω and integrate by parts

$$-\iiint_\Omega \nabla\cdot(e^{xyz}\nabla u)v\,dV = \iiint_\Omega (x+2y+3z)v\,dV$$

$$-\iint_\Gamma e^{xyz}(\nabla u\cdot\vec{n})v\,dS + \iiint_\Omega e^{xyz}\nabla u\cdot\nabla v\,dV = \iiint_\Omega (x+2y+3z)v\,dV$$

$$-\iint_{\Gamma_1} e^{xyz}\underbrace{(\nabla u\cdot\vec{n})}_{?}\underbrace{v}_{=0}\,dS - \iint_{\Gamma_2}\underbrace{e^{xyz}(\nabla u\cdot\vec{n})}_{=e^{xy}\nabla u\cdot\vec{n}=2}v\,dS - \iint_{\Gamma_3}\underbrace{e^{xyz}(\nabla u\cdot n)}_{=3-u}v\,dS$$

$$+\iiint_\Omega e^{xyz}\nabla u\cdot\nabla v\,dV = \iiint_\Omega (x+2y+3z)v\,dV$$

Hence the weak formulation is:

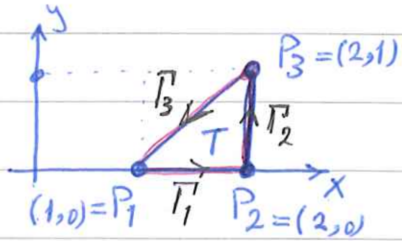
Find $u=u(x,y,z)$ such that $u=5$ on Γ_1 and

$$\iiint_\Omega e^{xyz}\nabla u\cdot\nabla v\,dV + \iint_{\Gamma_3} uv\,dS$$

$$= \iiint_\Omega (x+2y+3z)v\,dV + 2\iint_{\Gamma_2} v\,dS + 3\iint_{\Gamma_3} v\,dS$$

for every test function $v=v(x,y,z)$, such that $v=0$ on Γ_1

4.3



(a) $\varphi(x,y)=ax+by+c$ for some a,b,c

φ_1: $\varphi_1(x,y)=ax+by+c$

$$\begin{cases}\varphi_1(P_1)=1\\ \varphi_1(P_2)=0\\ \varphi_1(P_3)=0\end{cases}\Rightarrow\begin{cases}\varphi_1(1,0)=1\\ \varphi_1(2,0)=0\\ \varphi_1(2,1)=0\end{cases}\Rightarrow\begin{cases}a+c=1\\ 2a+c=0\\ 2a+b+c=0\end{cases}\Rightarrow\begin{cases}a=-1\\ c=2\\ b=0\end{cases}\Rightarrow \begin{array}{l}\varphi_1(x,y)=-x+2\\ \nabla\varphi_1(x,y)=\langle -1,0\rangle\end{array}$$

φ_2: $\varphi_2(x,y)=ax+by+c$

$$\begin{cases}\varphi_2(P_1)=0\\ \varphi_2(P_2)=1\\ \varphi_2(P_3)=0\end{cases}\Rightarrow\begin{cases}a+c=0\\ 2a+c=1\\ 2a+b+c=0\end{cases}\Rightarrow\begin{cases}a=1\\ c=-1\\ b=-1\end{cases}\Rightarrow \begin{array}{l}\varphi_2(x,y)=x-y-1\\ \nabla\varphi_2(x,y)=\langle 1,-1\rangle\end{array}$$

φ_3: $\varphi_3(x,y)=ax+by+c$

$$\begin{cases}\varphi_3(P_1)=0\\ \varphi_3(P_2)=0\\ \varphi_3(P_3)=1\end{cases}\Rightarrow\begin{cases}a+c=0\\ 2a+c=0\\ 2a+b+c=1\end{cases}\Rightarrow\begin{cases}a=0\\ c=0\\ b=1\end{cases}\Rightarrow \begin{array}{l}\varphi_3(x,y)=y\\ \nabla\varphi_3(x,y)=\langle 0,1\rangle\end{array}$$

(b) Γ_1: $x=1+t,\ y=0,\ 0\le t\le 1,$

$\vec{r}_1(t)=\langle 1+t,0\rangle,\ 0\le t\le 1 \Rightarrow \vec{r}_1{}'(t)=\langle 1,0\rangle \Rightarrow |\vec{r}_1{}'(t)|=1$

Γ_2: $x=2,\ y=t,\ 0\le t\le 1 \Rightarrow |\vec{r}_2{}'(t)|=|\langle 0,1\rangle|=1$

Γ_3: $x=2-t,\ y=1-t,\ 0\le t\le 1 \Rightarrow |\vec{r}_3{}'(t)|=|\langle -1,-1\rangle|=\sqrt{2}$

$$a_{11}=\iint_\Omega \underbrace{\nabla\varphi_1\cdot\nabla\varphi_1}_{=1}dA+\int_\Gamma \varphi_1\varphi_1\,ds=\overbrace{A(\Omega)}^{=\frac{1}{2}}+\int_\Gamma (2-x)^2ds$$

$$=\frac{1}{2}+\int_{\Gamma_1}(2-x)^2ds+\int_{\Gamma_2}(2-x)^2ds+\int_{\Gamma_3}(2-x)^2ds$$

$$=\frac{1}{2}+\int_0^1(1-t)^2\overbrace{|\vec{r}_1{}'(t)|}^{=1}dt+\int_0^1(0)^2\overbrace{|\vec{r}_2{}'(t)|}^{=1}dt+\int_0^1 t^2\overbrace{|\vec{r}_3{}'(t)|}^{=\sqrt{2}}dt$$

$$=\frac{1}{2}-\left[\frac{(1-t)^3}{3}\right]_0^1+\frac{\sqrt{2}}{3}=\frac{5+2\sqrt{2}}{6}$$

$$a_{22} = \iint_\Omega \underbrace{\nabla\varphi_2 \cdot \nabla\varphi_2}_{=2} \, dA + \int_\Gamma \varphi_2 \varphi_2 \, ds = 2A(\Omega) + \int_0^1 t^2 |\vec{r}_1{}'(t)| \, dt + \int_0^1 (1-t)^2 |\vec{r}_2{}'(t)| \, dt + \int_0^1 (0)^2 |\vec{r}_3{}'(t)| \, dt$$

$$= \frac{2}{2} + \frac{1}{3} + \frac{1}{3} + 0 = \frac{5}{3}$$

$$a_{33} = \iint_\Omega \underbrace{\nabla\varphi_3 \cdot \nabla\varphi_3}_{=1} \, dA + \int_\Gamma \varphi_3 \varphi_3 \, ds = A(\Omega) + \int_0^1 (0)^2 \underbrace{|\vec{r}_1{}'(t)|}_{1} \, dt + \int_0^1 t^2 \underbrace{|\vec{r}_2{}'(t)|}_{1} \, dt + \int_0^1 (1-t)^2 \underbrace{|\vec{r}_3{}'(t)|}_{\sqrt{2}} \, dt$$

$$= \frac{1}{2} + 0 + \frac{1}{3} + \frac{\sqrt{2}}{3} = \frac{5+2\sqrt{2}}{6}$$

$$a_{12} = a_{21} = \iint_\Omega \nabla\varphi_1 \cdot \nabla\varphi_2 \, dA + \int_\Gamma \varphi_1 \varphi_2 \, ds = -A(\Omega) + \int_0^1 (1-t)\,t\,(1) \, dt + \int_0^1 (0)(1-t)(1) \, dt + \int_0^1 t\,(0)\sqrt{2} \, dt$$

$$= -\frac{1}{2} + \frac{1}{6} = \frac{-1}{3}$$

$$a_{13} = a_{31} = \iint_\Omega \underbrace{\nabla\varphi_1 \cdot \nabla\varphi_3}_{=0} \, dA + \int_\Gamma \varphi_1 \varphi_3 \, ds = 0 + \int_0^1 (1-t)(0)(1) \, dt + \int_0^1 (0)\,t\,(1) \, dt + \int_0^1 t(1-t)\sqrt{2} \, dt$$

$$= \frac{\sqrt{2}}{6}$$

$$a_{23} = a_{32} = \iint_\Omega \underbrace{\nabla\varphi_2 \cdot \nabla\varphi_3}_{=-1} \, dA + \int_\Gamma \varphi_2 \varphi_3 \, ds$$

$$= -\frac{1}{2} + \int_0^1 t\,(0)(1) \, dt + \int_0^1 (1-t)\,t\,(1) \, dt + \int_0^1 (0)(1-t)(\sqrt{2}) \, dt$$

$$= -\frac{1}{2} + \frac{1}{6} = \frac{-1}{3}$$

So the stiffness matrix is

$$A = \begin{bmatrix} \frac{5+2\sqrt{2}}{6} & -\frac{1}{3} & \frac{\sqrt{2}}{6} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{6} & -\frac{1}{3} & \frac{5+2\sqrt{2}}{6} \end{bmatrix}$$

(c) $m_{11} = \iint_\Omega \varphi_1 \varphi_1 \, dA = \iint_\Omega (2-x)^2 dA = \int_1^2 \int_0^{x-1} (2-x)^2 dy\, dx$

$$= \int_1^2 (2-x)^2 (x-1)\, dx = \int_1^2 (x^3 - 5x^2 + 8x - 4)\, dx = \frac{1}{12}$$

$m_{22} = \iint_\Omega \varphi_2 \varphi_2 \, dA = \int_1^2 \int_0^{x-1} (x-y-1)^2 dy\, dx = \int_1^2 \left[\frac{-(x-y-1)^3}{3} \right]_{y=0}^{y=x-1} dx$

$$= \int_1^2 \frac{(x-1)^3}{3} dx = \frac{(x-1)^4}{12} \Big]_{x=1}^{2} = \frac{1}{12}$$

$m_{33} = \iint_\Omega \varphi_3 \varphi_3 \, dA = \int_1^2 \int_0^{x-1} y^2 dy\, dx = \int_1^2 \frac{(x-1)^3}{3} dx = \frac{1}{12}$

$m_{12} = m_{21} = \iint_\Omega \varphi_1 \varphi_2 dA = \int_1^2 \int_0^{x-1} (2-x)(x-y-1)\, dy\, dx$

$$= \int_1^2 \int_0^{x-1} (-x^2 + 3x + xy - 2y - 2)\, dy\, dx = \int_1^2 \left(\frac{-x^3}{2} + 2x^2 - \frac{5}{2}x + 1 \right) dx$$

$$= \frac{1}{24}$$

$m_{13} = m_{31} = \iint_\Omega \varphi_1 \varphi_3 dA = \int_1^2 \int_0^{x-1} (2-x)\, y\, dy\, dx = \int_1^2 \frac{(2-x)(x-1)^2}{2} dx$

$$= \frac{1}{24}$$

$m_{23} = m_{32} = \iint_\Omega \varphi_2 \varphi_3 \, dA = \int_1^2 \int_0^{x-1} (x-y-1)\, y\, dy\, dx = \frac{1}{24}$

So the mass matrix is

$$M = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{bmatrix}$$