Here is an improved version of Theorem 6.4.

Theorem 1. *The eigenfunctions* $\{\varphi_j\}_{j=1}^{\infty}$ *of (6.5) form an orthonormal basis for* L_2. *The series* $\sum_{j=1}^{\infty}\lambda_j(v,\varphi_j)^2$ *is convergent if and only if* $v \in H_0^1$. *Moreover,*

$$\|\nabla v\|^2 = a(v,v) = \sum_{j=1}^{\infty}\lambda_j(v,\varphi_j)^2, \quad \textit{for all } v \in H_0^1. \tag{1}$$

Proof. By our above discussion it follows that for the first statement it suffices to show (6.13) for all v in H_0^1, which is a dense subspace of L_2. We shall demonstrate that

$$\Big\| v - \sum_{j=1}^{N}(v,\varphi_j)\varphi_j \Big\| \le C\lambda_{N+1}^{-1/2}, \quad \text{for all } v \in H_0^1, \tag{2}$$

which then implies (6.13) in view of Theorem 6.3.

To prove (2), set $v_N = \sum_{j=1}^{N}(v,\varphi_j)\varphi_j$ and $r_N = v - v_N$. Then $(r_N,\varphi_j) = 0$ for $j = 1,\dots,N$, so that

$$\frac{\|\nabla r_N\|^2}{\|r_N\|^2} \ge \inf\Big\{\|\nabla v\|^2 : v \in H_0^1, \|v\| = 1, (v,\varphi_j) = 0, j = 1,\dots,N\Big\} = \lambda_{N+1},$$

and hence

$$\|r_N\| \le \lambda_{N+1}^{-1/2}\|\nabla r_N\|.$$

It now suffices to show that the sequence $\|\nabla r_N\|$ is bounded. We first recall from Theorem 6.1 that $a(\varphi_i,\varphi_j) = 0$ for $i \ne j$, so that $a(r_N, v_N) = 0$. Hence $a(v,v) = a(v_N,v_N) + 2a(v_N,r_N) + a(r_N,r_N) = a(v_N,v_N) + a(r_N,r_N)$ and

$$\|\nabla r_N\|^2 = a(r_N,r_N) = a(v,v) - a(v_N,v_N) \le a(v,v) = \|\nabla v\|^2,$$

which completes the proof of (2).

For the proof of the second statement, we first note that, for $v \in H_0^1$,

$$\sum_{j=1}^{N}\lambda_j(v,\varphi_j)^2 = a(v_N,v_N) = a(v,v) - a(r_N,r_N) \le a(v,v),$$

and we conclude that $\sum_{j=1}^{\infty}\lambda_j(v,\varphi_j)^2 < \infty$. Conversely, we assume that $v \in L_2$ and $\sum_{j=1}^{\infty}\lambda_j(v,\varphi_j)^2 < \infty$. We already know that $v_N \to v$ in L_2 as $N \to \infty$. To obtain convergence in H^1 we note that, with $M > N$,

$$\alpha\|v_N - v_M\|_1^2 \le \|\nabla(v_N - v_M)\|^2 = \sum_{j=N+1}^{M}\lambda_j(v,\varphi_j)^2 \to 0 \text{ as } N \to \infty.$$

Hence, v_N is a Cauchy sequence in H^1 and converges to a limit in H^1. Clearly, this limit is the same as v. By the trace theorem (Theorem A.4) v_N is also a Cauchy sequence in $L_2(\Gamma)$, and since $v_N = 0$ on Γ we conclude that $v = 0$ on Γ. Hence, $v \in H_0^1$. Finally, (1) is obtained by letting $N \to \infty$ in $a(v_N,v_N) = \sum_{j=1}^{N}\lambda_j(v,\varphi_j)^2$. $\square$

Here is an improved version of Theorem 13.1.

Theorem 2. *Let u_h and u be the solutions of (13.2) and (13.1). Then we have, for $t \geq 0$,*

$$\begin{aligned}
\|u_{h,t}(t) - u_t(t)\| &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u_t(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
\|u_h(t) - u(t)\| &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
|u_h(t) - u(t)|_1 &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_1 \, ds\Big).
\end{aligned}$$

Proof. Writing as usual

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho,$$

we may bound ρ and ρ_t as in the proof of Theorem 10.1 by

$$\|\rho(t)\| + h|\rho(t)|_1 \leq Ch^2\|u(t)\|_2, \quad \|\rho_t(t)\| \leq Ch^2\|u_t(t)\|_2. \tag{3}$$

For $\theta(t)$ we have, after a calculation analogous to that in (10.14),

$$(\theta_{tt}, \chi) + a(\theta, \chi) = -(\rho_{tt}, \chi), \quad \forall \chi \in S_h, \quad \text{for } t > 0. \tag{4}$$

Imitating the proof of Lemma 13.1, we choose $\chi = \theta_t$:

$$\tfrac{1}{2}\frac{d}{dt}\big(\|\theta_t\|^2 + |\theta|_1^2\big) \leq \|\rho_{tt}\| \, \|\theta_t\|.$$

After integration in t we obtain

$$\begin{aligned}
\|\theta_t(t)\|^2 + |\theta(t)|_1^2 &\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\| \, \|\theta_t\| \, \mathrm{d}s \\
&\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\| \, \mathrm{d}s \max_{s\in[0,t]} \|\theta_t\| \\
&\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2 + \tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2,
\end{aligned}$$

for $t \in [0, T]$. This implies

$$\tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2 \leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2$$

and hence

$$\|\theta_t(t)\|^2 + |\theta(t)|_1^2 \leq 2\|\theta_t(0)\|^2 + 2|\theta(0)|_1^2 + 4\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2,$$

for $t \in [0, T]$. In particular this holds with $t = T$ where T is arbitrary. Using also bounds for ρ_{tt} similar to (3), we obtain

$$\begin{aligned}
\|\theta_t(t)\| + \|\theta(t)\| &\leq C\Big(\|\theta_t(t)\| + |\theta(t)|_1\Big) \\
&\leq C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch^2\int_0^t \|u_{tt}\|_2 \, \mathrm{d}s,
\end{aligned}$$

and

$$|\theta(t)|_1 \leq C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch\int_0^t \|u_{tt}\|_1 \,\mathrm{d}s.$$

Together with the bounds in (3) this completes the proof. $\square$

We remark that the choices $v_h = R_h v$ and $w_h = R_h w$ in Theorem 2 give optimal order error estimates for all the three quantities considered, but that other optimal choices of v_h could cause a loss of one power of h, because of the gradient in the first term on the right. This can be avoided by a more refined argument. The regularity requirement on the exact solution can also be reduced.

`/stig`