Matematik Chalmers

Tentamen i
TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2011–05–24 fm V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Let $\mathcal{T} = \{\mathcal{T}_h\}_{h>0}$ be a *quasi-uniform* family of triangulations of a polygonal domain $\Omega \subset \mathbf{R}^2$. This means that there is a constant c such that

$$h_K \geq ch \quad \forall K \in \mathcal{T}_h,\ \forall \mathcal{T}_h \in \mathcal{T}, \quad \text{where } h_K = \operatorname{diam}(K),\ h = \max_{K \in \mathcal{T}_h} h_K.$$

Let S_h be the usual piecewise linear finite element space. Prove the *inverse inequality*

$$\|\nabla v_h\| \leq Ch^{-1}\|v_h\| \quad \forall v_h \in S_h$$

by the following steps.
(a) Assume for simplicity that K is obtained by translation and dilation of the unit size reference triangle $\hat{K}$ with corners $\hat{x} = (0,0), (0,1), (1,0)$, that is, $K = \{x : x = h_K\hat{x} + x_K,\ \hat{x} \in \hat{K}\}$ for some x_K. Let $\hat{v}_h(\hat{x}) = v(h_K\hat{x} + x_K)$. Show

$$\|v\|_{L_2(K)} = h_K\|\hat{v}\|_{L_2(\hat{K})}, \quad \|\nabla v\|_{L_2(K)} = \|\nabla\hat{v}\|_{L_2(\hat{K})} \quad \forall v \in H^1(K).$$

(b) Finish the proof by using that all norms on a finite-dimensional space are equivalent. In particular, $\|\cdot\|_{H^1(\hat{K})}$ and $\|\cdot\|_{L_2(\hat{K})}$ are equivalent on $\Pi_1(\hat{K})$, the space of all polynomials of degree ≤ 1.

2. State and prove the maximum principle for the elliptic operator $\mathcal{A}u = -\nabla\cdot(a\nabla u)$.

3. Consider the spatially semidiscrete finite element approximation of the heat equation

$$\begin{aligned} u_t - \Delta u &= f, && \text{in } \Omega \times \mathbf{R}_+,\\ u &= 0, && \text{on } \Gamma \times \mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

(a) Show the error estimate

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + Ch^2\Big\{\|v\|_2 + \int_0^t \|u_t(s)\|_2\,\mathrm{d}s\Big\}.$$

(b) Assume $f = 0$ and prove

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + Ch^2\|v\|_2.$$

4. Let $u_h(t) = E_h(t)v_h$ denote the spatially semidiscrete finite element solution of the heat equation in Problem 3 with $f = 0$ and initial approximation $v_h \in S_h$, $v_h \approx v$. Show that

(1) $$\|E_h(t)v_h\| \leq \|v_h\|, \quad t \geq 0,$$

(2) $$\|\mathrm{D}_tE_h(t)v_h\| \leq Ct^{-1}\|v_h\|, \quad t > 0,$$

(3) $$\|E_h(t)v_h - v_h\| \leq Ct^{1/2}|v_h|_1, \quad t \geq 0.$$

5. (a) What do we mean by a "Friedrichs system" ("symmetric hyperbolic system")?
(b) Write the pure initial value problem for the wave equation in $\mathbf{R}^2 \times \mathbf{R}_+$ as a Friedrichs system.
(c) Prove a stability estimate for the system in (b) or (a).
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TMA026/MMA430 Partial differential equations II
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1. (a) We have $\mathrm{d}x = h_K^2\,\mathrm{d}\hat{x}$ and $\dfrac{\partial v}{\partial x_j} = \dfrac{\partial \hat{v}}{\partial \hat{x}_j}\dfrac{\partial \hat{x}_j}{\partial x_j} = h_K^{-1}\dfrac{\partial \hat{v}}{\partial \hat{x}_j}$, so that

$$\|v\|_{L_2(K)}^2 = \int_K |v|^2\,\mathrm{d}x = \int_K |\hat{v}|^2 h_K^2\,\mathrm{d}\hat{x} = h_K^2\|\hat{v}\|_{L_2(\hat{K})}^2,$$

$$\|\nabla v\|_{L_2(K)}^2 = \sum_{j=1}^2 \int_K \Big|\frac{\partial v}{\partial x_j}\Big|^2\,\mathrm{d}x = \sum_{j=1}^2 \int_K h_K^{-2}\Big|\frac{\partial \hat{v}}{\partial \hat{x}_j}\Big|^2 h_K^2\,\mathrm{d}\hat{x} = \|\nabla\hat{v}\|_{L_2(\hat{K})}^2.$$

(b) The equivalence of $\|\cdot\|_{H^1(\hat{K})}$ and $\|\cdot\|_{L_2(\hat{K})}$ on $\Pi_1(\hat{K})$ implies that

$$\|\nabla v\|_{L_2(\hat{K})}^2 \le \|v\|_{L_2(\hat{K})}^2 + \|\nabla v\|_{L_2(\hat{K})}^2 = \|v\|_{H^1(\hat{K})}^2 \le C\|v\|_{L_2(\hat{K})}^2 \quad \forall v \in \Pi_1(\hat{K}),$$

Let $v_h \in S_h$. Then $\hat{v}_h|_{\hat{K}} \in \Pi_1(\hat{K})$ and using also the assumption $h_K \ge ch$ we obtain

$$\begin{aligned}\|\nabla v_h\|^2 &= \sum_{K\in\mathcal{T}_h}\|\nabla v_h\|_{L_2(K)}^2 = \sum_{K\in\mathcal{T}_h}\|\nabla\hat{v}_h\|_{L_2(\hat{K})}^2 \le C\sum_{K\in\mathcal{T}_h}\|\hat{v}_h\|_{L_2(\hat{K})}^2\\ &= C\sum_{K\in\mathcal{T}_h} h_K^{-2}\|v_h\|_{L_2(K)}^2 \le Cc^{-2}h^{-2}\sum_{K\in\mathcal{T}_h}\|v_h\|_{L_2(K)}^2 = Ch^{-2}\|v_h\|^2\end{aligned}$$

with a new C.

2. See Chapter 3.

3. (a) See Theorem 10.1.
(b) See the hint for Problem 10.4 (a) in the book.

4. (a) The finite element problem is

$$\begin{aligned}&u_h(t) \in S_h,\ u_h(0) = v_h,\\ &(u_{h,t},\chi) + (\nabla u_h, \nabla\chi) = 0 \quad \forall\chi \in S_h,\ t > 0.\end{aligned} \tag{4}$$

Take $\chi = u_h(t)$:

$$\frac{1}{2}\mathrm{D}_t\|u_h\|^2 + \|\nabla u_h\|^2 = 0$$

so that

$$\frac{1}{2}\|u_h(t)\|^2 + \int_0^t \|\nabla u_h(s)\|^2\,\mathrm{d}s = \frac{1}{2}\|v_h\|^2. \tag{5}$$

This proves (1).
(b) Differentiate the finite element problem:

$$(u_{h,tt},\chi) + (\nabla u_{h,t}, \nabla\chi) = 0 \quad \forall\chi \in S_h,\ t > 0.$$

Take $\chi = u_{h,t}(t)$:

$$\frac{1}{2}\mathrm{D}_t\|u_{h,t}\|^2 + \|\nabla u_{h,t}\|^2 = 0$$

Multiply by t^2:

$$\frac{1}{2}\mathrm{D}_t(t^2\|u_{h,t}\|^2) + t^2\|\nabla u_{h,t}\|^2 = t\|u_{h,t}\|^2$$

so that

$$\frac{1}{2}t^2\|u_{h,t}(t)\|^2 + \int_0^t s^2\|\nabla u_{h,t}(s)\|^2\,\mathrm{d}s = \int_0^t s\|u_{h,t}(s)\|^2\,\mathrm{d}s$$

To bound the righ-hand side we we take $\chi = u_{h,t}(t)$ in (4) and multiply by t:

$$t\|u_{h,t}\|^2 + \frac{1}{2}\mathrm{D}_t(t\|\nabla u_h\|^2) = \frac{1}{2}\|\nabla u_h\|^2,$$

so that, in view of (5),

$$\int_0^t s\|u_{h,t}(s)\|^2\,\mathrm{d}s + \frac{1}{2}t\|\nabla u_h(t)\|^2 = \frac{1}{2}\int_0^t \|\nabla u_h(s)\|^2\,\mathrm{d}s \le \frac{1}{4}\|v_h\|^2.$$

Therefore:

$$\frac{1}{2}t^2\|u_{h,t}(t)\|^2 \le \int_0^t s\|u_{h,t}(s)\|^2\,\mathrm{d}s \le \frac{1}{4}\|v_h\|^2.$$

This proves (2) (with $C = 1/\sqrt{2}$).
(c) To prove (3) we note

$$\|u_h(t) - v_h\|^2 = \Big\|\int_0^t u_{h,t}(s)\,\mathrm{d}s\Big\|^2 \le \Big(\int_0^t \|u_{h,t}(s)\|\,\mathrm{d}s\Big)^2 \le t\int_0^t \|u_{h,t}(s)\|^2\,\mathrm{d}s,$$

where we have

$$\int_0^t \|u_{h,t}(s)\|^2\,\mathrm{d}s \le \frac{1}{2}\|\nabla v_h\|^2 = \frac{1}{2}|v_h|_1^2$$

by taking $\chi = u_{h,t}(t)$ in (4). This proves (3) (with $C = 1/\sqrt{2}$).

5. (a) See Section 11.4, page 178.
(b) Set $u_1 = u_t$, $u_2 = u_{x_1}$, $u_3 = u_{x_2}$. Then

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= u_{tt} = u_{x_1x_1} + u_{x_2x_2} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2}\\
\frac{\partial u_2}{\partial t} &= u_{x_1t} = \frac{\partial u_1}{\partial x_1}\\
\frac{\partial u_3}{\partial t} &= u_{x_2t} = \frac{\partial u_1}{\partial x_2}
\end{aligned}$$

that is

$$\frac{\partial}{\partial t}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\frac{\partial}{\partial x_1}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\frac{\partial}{\partial x_2}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

or

$$\begin{aligned}
&\frac{\partial U}{\partial t} - \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\frac{\partial U}{\partial x_1} - \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\frac{\partial U}{\partial x_2} = 0\\
&\big[U(0)\big] = \begin{bmatrix}w\\v_{x_1}\\v_{x_2}\end{bmatrix}.
\end{aligned}$$

(c) See Theorem 11.5. In the particular case of part (b) above we have $B = 0$, $\tilde{B} = B - \frac{1}{2}\sum_{j=1}^{2} \partial A_j/\partial x_j = 0$, so that

$$
\begin{aligned}
&\frac{1}{2}\mathrm{D}_t\|U\|^2 = 0,\\
&\frac{1}{2}\|U(t)\|^2 = \frac{1}{2}\|U(0)\|^2,\\
&\frac{1}{2}\|u_1(t)\|^2 + \frac{1}{2}\|u_{x_1}(t)\|^2 + \frac{1}{2}\|u_{x_2}(t)\|^2 = \frac{1}{2}\|w\|^2 + \frac{1}{2}\|v_{x_1}\|^2 + \frac{1}{2}\|v_{x_2}\|^2,
\end{aligned}
$$

which means that the usual energy is conserved.

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