Matematik Chalmers

Tentamen i
TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2011–08–26 fm V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Give a weak formulation and prove existence and uniqueness of solutions to the boundary value problem (under suitable assumptions)

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f, && \text{in } \Omega,\\ u &= g, && \text{on } \Gamma. \end{aligned}$$

2. Let u be a solution of the initial-boundary value problem

$$\begin{aligned} u_t - \nabla\cdot(a(x)\nabla u) &= 0, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= 0, && \text{in } \Gamma\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

Make the usual assumptions about the coefficient a and show that

$$\begin{aligned} \|u(t)\|_1 &\le C\|v\|_1, && t\ge 0,\\ \|u(t)\|_2 &\le C\|v\|_2, && t\ge 0,\\ |u(t)|_1 &\le Ct^{-1/2}\|v\|, && t>0. \end{aligned}$$

3. Consider the spatially semidiscrete finite element approximation of the heat equation

$$\begin{aligned} u_t - \Delta u &= f, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= 0, && \text{on } \Gamma\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

Show the error estimate

$$\|u_h(t)-u(t)\| \le \|v_h - v\| + Ch^2\Big\{\|v\|_2 + \int_0^t \|u_t(s)\|_2\,\mathrm{d}s\Big\}.$$

4. (a) Let Ω be the unit square in $\mathbf{R}^2$ with boundary Γ. Prove the inequality

$$\|v\|_{L_1(\Gamma)} \le C\|v\|_{W_1^1(\Omega)}, \quad \forall v\in\mathcal{C}^1(\bar\Omega).$$

Here $\|v\|_{L_1(\Gamma)} = \int_\Gamma |v|\,\mathrm{d}s$ and $\|v\|_{W_1^1(\Omega)} = \|v\|_{L_1(\Omega)} + \|\nabla v\|_{L_1(\Omega)}$.
(b) Use this to construct a trace operator γ.

5. Consider the initial-boundary value problem

$$\begin{aligned} u_t + a\cdot\nabla u + a_0 u &= f, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= g, && \text{in } \Gamma_-\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

(a) Present the method of characteristics for this problem.
(b) Assume $a_0(x) - \frac{1}{2}\nabla\cdot a(x) \ge \alpha > 0$ and show the energy estimate

$$\|u(t)\|^2 + \int_0^t\int_{\Gamma_+} u^2\, n\cdot a\,\mathrm{d}s\,\mathrm{d}t + \alpha\int_0^t \|u\|^2\,\mathrm{d}t \le \|v\|^2 + \int_0^t \|f\|^2\,\mathrm{d}t + \int_0^t\int_{\Gamma_-} g^2|n\cdot a|\,\mathrm{d}s\,\mathrm{d}t.$$

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TMA026/MMA430 Partial differential equations II
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1. See the book, Section 3.5.

2. The weak formulation is

$$
\begin{aligned}
&u(t) \in H_0^1(\Omega),\ u(0) = v, \\
&(u_t, \varphi) + a(u, \varphi) = 0, \quad \forall \varphi \in H_0^1(\Omega),
\end{aligned} \tag{1}
$$

where $a(\cdot,\cdot) = (a\nabla\cdot, \nabla\cdot)$. As usual we assume: $0 < a_0 \le a(x) \le a_1$ for all $x \in \bar{\Omega}$, so that

$$
\begin{aligned}
a(v, v) &\ge a_0 |v|_1^2, \quad \forall v \in H_0^1, \\
|a(v, w)| &\le a_1 |v|_1 \, |w|_1, \quad \forall v, w \in H_0^1.
\end{aligned}
$$

Here $|v|_1 = \|\nabla v\|$.

(a) Take $\varphi = u$:

$$
\begin{aligned}
&(u_t, u) + a(u, u) = 0, \\
&\frac{1}{2}\frac{d}{dt}\|u\|^2 + a(u, u) = 0, \\
&\|u(t)\|^2 + 2\int_0^t a(u, u)\, ds = \|v\|^2, && (2) \\
&\|u(t)\| \le \|v\|. && (3)
\end{aligned}
$$

(b) Take $\varphi = u_t$:

$$
\begin{aligned}
&\|u_t\|^2 + a(u, u_t) = 0, \\
&\|u_t\|^2 + \frac{1}{2}\frac{d}{dt} a(u, u) = 0, \qquad \text{[because } a(\cdot,\cdot) \text{ is symmetric]} && (4) \\
&2\int_0^t \|u_t\|^2\, ds + a(u(t), u(t)) = a(v, v), \\
&a(u(t), u(t)) \le a(v, v), \\
&a_0 \|\nabla u(t)\|^2 \le a(u(t), u(t)) \le a(v, v) && (5) \\
&\qquad \le a_1 \|\nabla v\|^2 + \|v\|^2 \le (a_1 + c)\|\nabla v\|^2, \\
&\|\nabla u(t)\| \le C \|\nabla v\|.
\end{aligned}
$$

Together with (3) this proves

$$\|u(t)\|_1 \le \|v\|_1.$$

(c) Differentiate (1) with respect to t and then take $\varphi = u_t$:

$$
\begin{aligned}
&(u_{tt}, u_t) + a(u_t, u_t) = 0, \\
&\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + a(u_t, u_t) = 0, \\
&\|u_t(t)\|^2 + 2\int_0^t a(u_t, u_t)\, ds = \|u_t(0)\|^2, \\
&\|u_t(t)\| \le \|u_t(0)\|.
\end{aligned}
$$

Now $u_t = \nabla \cdot (a\nabla u) = a\Delta u + \nabla a \cdot \nabla u$. Therefore $\|u_t(0)\| = \|a\Delta v + \nabla a \cdot \nabla v\| \le C\|v\|_2$, so that

$$\|u_t(t)\| \le C\|v\|_2.$$

Also, using elliptic regularity and the previous estimates of $\|u_t(t)\|$ and $\|u(t)\|_1$,

$$\|u(t)\|_2 \le C\|\Delta u\| \le C\|a\Delta u\| = C\|u_t - \nabla a \cdot \nabla u\| \le C(\|u_t(t)\| + \|u(t)\|_1) \le C\|v\|_2.$$

(d) Multiply (4) by t:

$$\begin{aligned}
&t\|u_t\|^2 + \frac{1}{2}t\frac{d}{dt}a(u,u) = 0,\\
&2t\|u_t\|^2 + \frac{d}{dt}\big(ta(u,u)\big) = a(u,u),\\
&2\int_0^t s\|u_t\|^2\,ds + ta(u(t),u(t)) = \int_0^t a(u,u)\,ds,\\
&a_0 t\|\nabla u(t)\|^2 \le ta(u(t),u(t)) \le \int_0^t a(u,u)\,ds \le \frac{1}{2}\|v\|^2, \quad \text{[by (2)]}\\
&\|\nabla u(t)\| \le Ct^{-1/2}\|v\|.
\end{aligned}$$

3. See Theorem 10.1.

4. (a) Start with

$$v(0,x_2) = v(x_1,x_2) - \int_0^{x_1} \frac{\partial v}{\partial x_1}(y,x_2)\,dy.$$

(b) In a similar way as in the trace theorem in the book we can define $\gamma : W_1^1(\Omega) \to L_1(\Gamma)$.

5. (a) The characteristics $x = x(s)$, $t = t(s)$ are given by

$$\begin{aligned}
\frac{dx}{ds} &= a(x(s),t(s)),\\
\frac{dt}{ds} &= t.
\end{aligned}$$

Hence $t = s + C$, choose $C = 0$ so that $s = 0$ at the initial time. Then $t = s$ and the first equation becomes

$$\frac{dx}{dt} = a(x(t),t).$$

Then $w(t) = u(x(t),t)$ satisfies the equation

$$\frac{dw(t)}{dt} + a_0(x(t),t)w(t) = f(x(t),t). \tag{6}$$

To find the solution at $(\bar{x},\bar{t})$ we follow the characteristic thru $(\bar{x},\bar{t})$ backwards until we hit $t = 0$ at x_0 or hit Γ_- at (x_0,t_0). Then we solve (6) with the initial condition $w(0) = v(x_0)$ or $w(t_0) = g(x_0)$.

(b) Multiply by $u(t)$ and integrate by parts.

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