Matematik Chalmers

Tentamen i
TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2012–05–22 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Prove the inequality

$$\|v\|^2_{L_2(\Omega)} \le C\Big(\|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Gamma)}\Big) \quad \forall v \in H^1(\Omega). \tag{1}$$

Here Ω is a bounded domain in $\mathbf{R}^d$ with smooth boundary Γ. Hint: Integrate by parts in the identity $\int_\Omega v^2\,dx = \int_\Omega v^2 \Delta\phi\,dx$, where $\phi(x) = \frac{1}{2d}|x|^2$.

2. (a) Formulate a finite element method for the problem

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f, && \text{in } \Omega,\\ a\frac{\partial u}{\partial n} + h(u-u_0) &= g, && \text{on } \Gamma. \end{aligned}$$

Assume $a(x) \ge a_0 > 0$, $h(x) \ge h_0 > 0$, $\|a\|_{\mathcal{C}} < \infty$, $\|h\|_{\mathcal{C}} < \infty$ and $\Omega \subset \mathbf{R}^2$ a polygonal domain.
(b) Prove an error estimate in the H^1-norm (under suitable assumptions). Hint: use problem 1.
(c) Prove an error estimate in the L_2-norm (under suitable assumptions).

3. Let u be a solution of

$$\begin{aligned} u_t - \nabla\cdot(a\nabla u) &= f, && \text{in } \Omega\times\mathbf{R}_+,\\ a\frac{\partial u}{\partial n} + h(u-u_0) &= g, && \text{on } \Gamma\times\mathbf{R}_+,\\ u(\cdot,0) &= v && \text{in } \Omega. \end{aligned}$$

Assume that a, h are as in the previous problem. Show that

$$\|u(t)\|^2 + \int_0^t \|u\|_1^2\,\mathrm{d}s \le C\Big(\|v\|^2 + \int_0^t \big(\|f\|^2 + \|u_0\|^2_{L_2(\Gamma)} + \|g\|^2_{L_2(\Gamma)}\big)\,\mathrm{d}s\Big).$$

4. State and prove the maximum principle for the heat operator $\partial u/\partial t - \Delta u$.

5. (a) What do we mean by a "Friedrichs system" ("symmetric hyperbolic system")?
(b) Write the pure initial value problem for the wave equation in $\mathbf{R}^2\times\mathbf{R}_+$ as a Friedrichs system.
(c) Prove a stability estimate for the system in (b).

PDE II 2012-05-22 Solutions

1) $\int_\Omega v^2 dx = \int_\Omega v^2 \underbrace{\Delta\phi}_{=1} dx = \int_\Gamma v^2 n\cdot\nabla\phi \, ds - \int_\Omega \underbrace{\nabla v^2}_{=2v\nabla v}\cdot\nabla\phi \, dx$

$\leq \|\phi\|_{\mathcal{C}^1} \int_\Gamma v^2 ds + 2\|\phi\|_{\mathcal{C}^1} \int_\Omega |v||\nabla v| dx$

$\leq C\|v\|_\Gamma^2 + 2C\|v\|\,\|\nabla v\|$ $\quad \left[C = \|\phi\|_{\mathcal{C}^1} \right]$

$\leq C\|v\|_\Gamma^2 + \frac{1}{2}\|v\|^2 + \frac{1}{2}(2C)^2\|\nabla v\|^2$

$\frac{1}{2}\|v\|^2 \leq C\|v\|_\Gamma^2 + 2C^2\|\nabla v\|^2$

$\|v\| \leq C\left(\|v\|_\Gamma^2 + \|\nabla v\|^2\right)$ (new c)

2) a) $\hat{S}_h = \{ v_h \in \mathcal{C}(\bar\Omega) : v_h|_K \in \Pi_1 \}$ (no boundary condition)

$\hat{S}_h \subset H^1 = H^1(\Omega)$

$$\begin{cases} u_h \in \hat{S}_h \\ a(u_h, \chi) = L(\chi) \quad \forall \chi \in \hat{S}_h \end{cases}$$

$a(w,v) = (a\nabla v, \nabla w) + (h v, w)_\Gamma$

$L(v) = (f, v) + (h u_0 + g, v)_\Gamma$

Assume:

$a(x) \geq a_0 > 0$, $h(x) \geq h_0 > 0$, $\|a\|_{\mathcal{C}} < \infty$, $\|h\|_{\mathcal{C}} < \infty$

$u_0, g \in L_2(\Gamma)$

Then, with problem 1, we show that

$a(v,v) \geq c\|v\|_1^2$

$a(v,w) \leq C\|v\|_1\|w\|_1 \quad \forall v, w \in H^1$

$L(v) \leq C\|v\|_1$

So $a(\cdot,\cdot)$ is a scalar product equivalent with the standard H^1-scalar product.

$\|v\|_a = \sqrt{a(v,v)}$.

b) (*) $a(u_h - u, \chi) = 0 \quad \forall \chi \in S_h$ (orthogonality)

Hence $\|u_h - u\|_a = \inf_{v_h \in S_h} \|u - v_h\|_a$

Norm equivalence and interpol. error:

$$\|u_h - u\|_1 \le C \inf_{v_h} \|u - v_h\|_1$$

$$\le C \|u - I_h u\|_1 \le$$

$$= C \sqrt{\underbrace{\|u_h - I_h u\|^2}_{\le C h^4 |u|_2^2} + \underbrace{|u_h - I_h u|_1^2}_{C h^2 |u|_2^2}}$$

$$\le C \sqrt{h^4 + h^2}\, |u|_2 \le C h |u|_2$$

c) adjoint problem: $e = u_h - u$

$$\begin{cases} -\nabla \cdot (a \nabla \phi) = e & \text{in } \Omega \\ a \frac{\partial \phi}{\partial n} + h u = 0 & \text{on } \Gamma \end{cases}$$

Weak form: $\begin{cases} \phi \in H^1 \\ a(w, \phi) = (w, e) \quad \forall w \in H^1 \end{cases}$

Take $w = e$ and use (*):

$$\|e\|^2 = (e, e) = a(e, \phi) \overset{(*)}{=} a(e, \phi - I_h \phi)$$

$$\le C \|e\|_1 \|\phi - I_h \phi\|_1 \le C \|e\|_1 \cdot C h \|\phi\|_2$$

$$\le C \|e\|_1 \cdot C h \|e\|$$

$$\|e\| \le C h \|e\|_1 \le C h^2 \|u\|_2$$

3) $\begin{cases} u(t) \in H^1, \quad u(0) = v \\ (u_t, \varphi) + a(u, \varphi) = (f, \varphi) + (h u_0 + g, \varphi)_\Gamma \quad \forall \varphi \in H^1, t > 0 \end{cases}$

with $a(\cdot,\cdot)$ as in problem 2.

$\varphi = u(t)$: $(u_t, u) + \underbrace{a(u, u)}_{\geq c \|u\|_1^2} = (f, u) + (h u_0 + g, u)$

$$\frac{1}{2} D_t \|u\|^2 + c \|u\|_1^2 \leq \|f\| \underbrace{\|u\|}_{\leq \|u\|_1} + \underbrace{\|h\|_\infty}_{\leq C} (\|u_0\|_\Gamma + \|g\|_\Gamma) \underbrace{\|u\|_\Gamma}_{\leq C\|u\|_1}$$

(trace ineq.)

$$\leq (\|f\| + C(\|u_0\|_\Gamma + \|g\|_\Gamma)) \|u\|_1$$

$$\leq C(\|f\|^2 + \|u_0\|_\Gamma^2 + \|g\|_\Gamma^2) + \frac{c}{2} \|u\|_1^2$$

$$\frac{1}{2} D_t \|u\|^2 + \frac{1}{2} c \|u\|_1^2 \leq C(\|f\|^2 + \|u_0\|_\Gamma^2 + \|g\|_\Gamma^2)$$

$$\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds \leq C\left(\|v\|^2 + \int_0^t \|f\|^2 + \|u_0\|_\Gamma^2 + \|g\|_\Gamma^2 ds\right)$$

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4. See the book.

5. See the book.