Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2012–08–31 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. For the solution of the homogeneous heat equation

$$u_t - \Delta u = 0 \quad \text{in } \Omega \times \mathbf{R}_+; \qquad u = 0 \quad \text{on } \Gamma \times \mathbf{R}_+; \qquad u(\cdot,0) = v \quad \text{in } \Omega,$$

we have the bounds

$$\int_0^t \|\nabla u(s)\|^2 \, \mathrm{d}s \le C\|v\|^2, \qquad \|\nabla u(t)\|^2 \le Ct^{-1}\|v\|^2.$$

(a) Prove these by means of eigenfunction expansion.

(b) Prove these by means of the energy method. Find good values of the constants in both methods.

2. (a) Formulate a finite element method for the problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= f, && \text{in } \Omega, \\ u &= 0, && \text{on } \Gamma. \end{aligned}$$

Assume $a(x) \ge a_0 > 0$ and $\Omega \subset \mathbf{R}^2$ a polygonal domain.

(b) Prove an error estimate in the H^1-norm (under suitable assumptions).

3. State and prove the maximum principle for the heat operator $\partial u/\partial t - \Delta u$.

4. Consider the initial-boundary value problem

$$\begin{aligned} u_t + a \cdot \nabla u + a_0 u &= f, && \text{in } \Omega \times \mathbf{R}_+, \\ u &= g, && \text{in } \Gamma_- \times \mathbf{R}_+, \\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

(a) Present the method of characteristics for this problem.

(b) Prove an energy estimate for u under the assumption $a_0(x) - \frac{1}{2}\nabla \cdot a(x) \ge \alpha > 0$.

5. Consider the Navier-Stokes equations for the motion of an incompressible fluid: find a vector field $u = (u_1, u_2)$ and a scalar field p such that

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= f, && \text{in } \Omega \times \mathbf{R}_+, \\ \nabla \cdot u &= 0, && \text{in } \Omega \times \mathbf{R}_+, \\ u &= 0, && \text{on } \Gamma \times \mathbf{R}_+, \\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned} \tag{1}$$

Here $\Omega \subset \mathbf{R}^2$ and $f = f(x,t)$, $v = v(x)$ are given vector fields and we define $u \cdot \nabla = \sum_{i=1}^2 u_i \frac{\partial}{\partial x_i}$ so that $((u \cdot \nabla)u)_j = \sum_{i=1}^2 u_i \frac{\partial u_j}{\partial x_i}$. Let $(L_2)^2 = \{v = (v_1, v_2) : v_i \in L_2\}$ with scalar product $(u,v) = \int_\Omega u \cdot v \, dx$ and $(H_0^1)^2 = \{v = (v_1, v_2) : v_i \in H_0^1\}$ with norm $|v|_1^2 = \sum_{i=1}^2 \sum_{j=1}^2 \|\partial v_i/\partial x_j\|^2$.

(a) Show that, for all $p \in H^1,\ u, v, w \in (H_0^1)^2$,

$$(\nabla p, u) = -(p, \nabla \cdot u), \qquad ((u \cdot \nabla)v, w) = -((\nabla \cdot u)v, w) - (v, (u \cdot \nabla)w).$$

(b) Assume that u, p satisfy (1). Show that $((u \cdot \nabla)u, u) = 0$ and $(\nabla p, u) = 0$. Prove an energy estimate for u.

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1. (a) Eigenfunction expansion:

$$u(t)=\sum_{j=1}^{\infty}\mathrm{e}^{-\lambda_j t}\hat{v}_j\varphi_j.$$

Parseval:

$$\|\nabla u(t)\|^2=\sum_{j=1}^{\infty}\lambda_j\mathrm{e}^{-2\lambda_j t}\hat{v}_j^2=\tfrac12 t^{-1}\sum_{j=1}^{\infty}(2\lambda_j t)\mathrm{e}^{-2\lambda_j t}\hat{v}_j^2\le\tfrac{1}{2\mathrm{e}}t^{-1}\sum_{j=1}^{\infty}\hat{v}_j^2=\tfrac{1}{2\mathrm{e}}t^{-1}\|v\|^2,$$

because $\max_{x\ge0}x\mathrm{e}^{-x}=\mathrm{e}^{-1}$ is attained for $x=1$. Parseval again:

$$\begin{aligned}\int_0^t\|\nabla u(s)\|^2\,\mathrm{d}s&=\int_0^t\sum_{j=1}^{\infty}\lambda_j\mathrm{e}^{-2\lambda_j s}\hat{v}_j^2\,\mathrm{d}s=\sum_{j=1}^{\infty}\int_0^t\lambda_j\mathrm{e}^{-2\lambda_j s}\,\mathrm{d}s\,\hat{v}_j^2\\&=\sum_{j=1}^{\infty}\Big[-\tfrac12\mathrm{e}^{-2\lambda_j s}\Big]_{s=0}^{t}\hat{v}_j^2=\tfrac12\sum_{j=1}^{\infty}\Big(1-\mathrm{e}^{-2\lambda_j t}\Big)\hat{v}_j^2\\&\le\tfrac12\sum_{j=1}^{\infty}\hat{v}_j^2=\tfrac12\|v\|^2.\end{aligned}$$

(b) Energy method, weak formulation:

$$u(t)\in H_0^1;\quad (u_t,\phi)+(\nabla u,\nabla\phi)=0\quad\forall\phi\in H_0^1,\ t>0.$$

Take $\phi=u(t)\in H_0^1$:

$$\begin{aligned}&(u_t,u)+(\nabla u,\nabla u)=0,\\&\tfrac12\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\|\nabla u\|^2=0,\\&\tfrac12\|u(t)\|^2+\int_0^t\|\nabla u\|^2\,\mathrm{d}s=\tfrac12\|v\|^2,\\&\int_0^t\|\nabla u\|^2\,\mathrm{d}s\le\tfrac12\|v\|^2.\end{aligned}$$

Take $\phi=tu_t(t)\in H_0^1$:

$$\begin{aligned}&t(u_t,u_t)+t(\nabla u,\nabla u_t)=0,\\&t\|u_t\|^2+\tfrac12\frac{\mathrm{d}}{\mathrm{d}t}\big(t\|\nabla u\|^2\big)-\tfrac12\|\nabla u\|^2=0,\\&\int_0^t s\|u_t\|^2\,\mathrm{d}s+\tfrac12 t\|\nabla u(t)\|^2=\tfrac12\int_0^t\|\nabla u\|^2\,\mathrm{d}s,\end{aligned}$$

so that, by the first part,

$$\begin{aligned}&t\|\nabla u(t)\|^2\le\int_0^t\|\nabla u\|^2\,\mathrm{d}s\le\tfrac12\|v\|^2,\\&\|\nabla u(t)\|^2\le\tfrac12 t^{-1}\|v\|^2.\end{aligned}$$

2. See the book.

3. See the book.

4. (a) The characteristics $x = x(s)$, $t = t(s)$ are given by

$$\frac{dx}{ds} = a(x(s), t(s)),$$
$$\frac{dt}{ds} = t.$$

Hence $t = s + C$, choose $C = 0$ so that $s = 0$ at the initial time. Then $t = s$ and the first equation becomes

$$\frac{dx}{dt} = a(x(t), t).$$

Then $w(t) = u(x(t), t)$ satisfies the equation

$$\frac{dw(t)}{dt} + a_0(x(t), t)w(t) = f(x(t), t). \tag{2}$$

To find the solution at $(\bar{x}, \bar{t})$ we follow the characteristic thru $(\bar{x}, \bar{t})$ backwards until we hit $t = 0$ at x_0 or hit Γ_- at (x_0, t_0). Then we solve (2) with the initial condition $w(0) = v(x_0)$ or $w(t_0) = g(x_0)$.
(b) We assume $a_0(x) - \frac{1}{2}\nabla \cdot a(x) \geq \alpha > 0$. Then it is easy to show

$$\|u(t)\|^2 + \int_0^t \int_{\Gamma_+} u^2\, n \cdot a\, ds\, dt + \alpha \int_0^t \|u\|^2\, dt \leq \|v\|^2 + \int_0^t \|f\|^2\, dt + \int_0^t \int_{\Gamma_-} g^2 |n \cdot a|\, ds\, dt.$$

5. (a) Integrate by parts using Green's formula and use the fact that u, v, w are $= 0$ on Γ.
(b) That $((u \cdot \nabla)u, u) = 0$ and $(\nabla p, u) = 0$ follows directly from (a). Then the standard energy argument gives

$$\|u(t)\|^2 + \int_0^t |u|_1^2\, ds = \|v\|^2 + C \int_0^t \|f\|^2\, dt.$$

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