Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2013–01–14 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. For the solution of the homogeneous heat equation

$$u_t - \Delta u = 0 \quad \text{in } \Omega \times \mathbf{R}_+; \qquad u = 0 \quad \text{on } \Gamma \times \mathbf{R}_+; \qquad u(\cdot, 0) = v \quad \text{in } \Omega,$$

we have the bounds

$$\int_0^t \|\nabla u(s)\|^2 \, \mathrm{d}s \le C\|v\|^2, \qquad \|\nabla u(t)\|^2 \le Ct^{-1}\|v\|^2.$$

(a) Prove these by means of eigenfunction expansion.

(b) Prove these by means of the energy method. Find good values of the constants in both methods.

2. (a) Formulate a finite element method for the problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= f, \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \Gamma. \end{aligned}$$

Assume $a(x) \ge a_0 > 0$ and $\Omega \subset \mathbf{R}^2$ a polygonal domain.

(b) Prove an error estimate in the H^1-norm (under suitable assumptions).

3. State and prove the Trace Theorem.

4. Consider solutions of the equation

$$u_{tt} - \Delta u = 0, \qquad x \in \Omega, \ t > 0,$$

with $u = 0$ on Γ for $t > 0$. Show that the energy $E(t) = \int_\Omega (|u_t(x,t)|^2 + |\nabla u(x,t)|^2) \, \mathrm{d}x$ is independent of t by means of (a) the energy method, (b) the spectral method (eigenfunction expansion).

5. (a) What do we mean when we say that a system of the form $u_t + Au_x = 0$ is a "strictly hyperbolic system"?

(b) Consider the system of first order PDEs:

$$\begin{aligned} y_t + y_x - 3z_x &= 0, \\ z_t - 3y_x + z_x &= 0, \end{aligned} \qquad x \in \mathbf{R}, \ t > 0.$$

Show that it is a strictly hyperbolic system.

(c) Solve the system by the method of characteristics. Use the initial condition $y(x, 0) = \cos x$, $z(x, 0) = 0$.

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1. (a) Eigenfunction expansion:

$$u(t) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \hat{v}_j \varphi_j.$$

Parseval:

$$\|\nabla u(t)\|^2 = \sum_{j=1}^{\infty} \lambda_j \mathrm{e}^{-2\lambda_j t} \hat{v}_j^2 = \tfrac{1}{2} t^{-1} \sum_{j=1}^{\infty} (2\lambda_j t) \mathrm{e}^{-2\lambda_j t} \hat{v}_j^2 \leq \tfrac{1}{2\mathrm{e}} t^{-1} \sum_{j=1}^{\infty} \hat{v}_j^2 = \tfrac{1}{2\mathrm{e}} t^{-1} \|v\|^2,$$

because $\max_{x \geq 0} x\mathrm{e}^{-x} = \mathrm{e}^{-1}$ is attained for $x = 1$. Parseval again:

$$\begin{aligned}
\int_0^t \|\nabla u(s)\|^2 \,\mathrm{d}s &= \int_0^t \sum_{j=1}^{\infty} \lambda_j \mathrm{e}^{-2\lambda_j s} \hat{v}_j^2 \,\mathrm{d}s = \sum_{j=1}^{\infty} \int_0^t \lambda_j \mathrm{e}^{-2\lambda_j s} \,\mathrm{d}s \, \hat{v}_j^2 \\
&= \sum_{j=1}^{\infty} \Big[-\tfrac{1}{2} \mathrm{e}^{-2\lambda_j s} \Big]_{s=0}^{t} \hat{v}_j^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} \Big(1 - \mathrm{e}^{-2\lambda_j t} \Big) \hat{v}_j^2 \\
&\leq \tfrac{1}{2} \sum_{j=1}^{\infty} \hat{v}_j^2 = \tfrac{1}{2} \|v\|^2.
\end{aligned}$$

(b) Energy method, weak formulation:

$$u(t) \in H_0^1; \quad (u_t, \phi) + (\nabla u, \nabla \phi) = 0 \quad \forall \phi \in H_0^1, \ t > 0.$$

Take $\phi = u(t) \in H_0^1$:

$$\begin{aligned}
&(u_t, u) + (\nabla u, \nabla u) = 0, \\
&\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2 + \|\nabla u\|^2 = 0, \\
&\tfrac{1}{2} \|u(t)\|^2 + \int_0^t \|\nabla u\|^2 \,\mathrm{d}s = \tfrac{1}{2} \|v\|^2, \\
&\int_0^t \|\nabla u\|^2 \,\mathrm{d}s \leq \tfrac{1}{2} \|v\|^2.
\end{aligned}$$

Take $\phi = t u_t(t) \in H_0^1$:

$$\begin{aligned}
&t(u_t, u_t) + t(\nabla u, \nabla u_t) = 0, \\
&t\|u_t\|^2 + \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(t\|\nabla u\|^2 \big) - \tfrac{1}{2} \|\nabla u\|^2 = 0, \\
&\int_0^t s\|u_t\|^2 \,\mathrm{d}s + \tfrac{1}{2} t \|\nabla u(t)\|^2 = \tfrac{1}{2} \int_0^t \|\nabla u\|^2 \,\mathrm{d}s,
\end{aligned}$$

so that, by the first part,

$$\begin{aligned}
&t\|\nabla u(t)\|^2 \leq \int_0^t \|\nabla u\|^2 \,\mathrm{d}s \leq \tfrac{1}{2} \|v\|^2, \\
&\|\nabla u(t)\|^2 \leq \tfrac{1}{2} t^{-1} \|v\|^2.
\end{aligned}$$

2. See the book.

3. See the book.

4. (a) This is Theorem 11.2.
(b) Let λ_j be the eigenvalues of $-\Delta$ and φ_j be the corresponding ON-basis of eigenfunctions. Inserting the Fourier series $u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\varphi_j(x)$ into the wave equation yields $\hat{u}_j(t) = a_j \cos(\sqrt{\lambda_j}t) + b_j \sin(\sqrt{\lambda_j}t)$, where a_j, b_j depend on the initial values. Parseval's relation implies

$$\begin{aligned}
E(t) &= \|u_t(t)\|^2 + \|\nabla u(t)\|^2 \\
&= \sum_{j=1}^{\infty} \Big(\hat{u}_j'(t)^2 + \lambda_j \hat{u}_j(t)^2 \Big) \\
&= \sum_{j=1}^{\infty} \lambda_j \Big((-a_j \sin(\sqrt{\lambda_j}t) + b_j \cos(\sqrt{\lambda_j}t))^2 + (a_j \cos(\sqrt{\lambda_j}t) + b_j \sin(\sqrt{\lambda_j}t))^2 \Big) \\
&= \sum_{j=1}^{\infty} \lambda_j (a_j^2 + b_j^2)(\sin^2(\sqrt{\lambda_j}t) + \cos^2(\sqrt{\lambda_j}t)) \\
&= \sum_{j=1}^{\infty} \lambda_j (a_j^2 + b_j^2)
\end{aligned}$$

which is independent of t.

5. (a) The system $u_t + Au_x = 0$ is strictly hyperbolic if the matrix A is symmetric with distinct eigenvalues.
(b) Systemet kan skrivas $u_t + Au_x = 0$ med $u = \begin{bmatrix} y \\ z \end{bmatrix}$ och $A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$. Diagonalisera A:

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^T A P = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, \quad w = P^T u.$$

Vi ser att matrisen A är symmetrisk med distinkta egenvärden.
(c) Systemet blir

$$w_t + \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} w_x = 0, \quad x \in \mathbf{R},\ t > 0; \qquad w(x,0) = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos x \\ \cos x \end{bmatrix}, \quad x \in \mathbf{R}.$$

De karakteristiska kurvorna genom punkten $(\bar{x}, \bar{t})$
är

$$x = x_1(t) = \bar{x} - 2(t - \bar{t}), \quad x = x_2(t) = \bar{x} + 4(t - \bar{t}).$$

Komponenten w_i är konstant längs $x = x_i(t)$:

$$w_1(\bar{x}, \bar{t}) = \frac{1}{\sqrt{2}} \cos(\bar{x} + 2\bar{t}), \quad w_2(\bar{x}, \bar{t}) = \frac{1}{\sqrt{2}} \cos(\bar{x} - 4\bar{t}),$$

och lösningen blir

$$u(x,t) = Pw(x,t) = \frac{1}{2} \begin{bmatrix} \cos(x + 2t) + \cos(x - 4t) \\ \cos(x + 2t) - \cos(x - 4t) \end{bmatrix}.$$

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